Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II

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Consider the 3-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory associated with a compact Lie group G_c and its quaternionic representation $\mathbf{M}$. Physicists study its Coulomb branch, which is a noncompact hyper-Kähler manifold with an SU(2)-action, possibly with singularities. We give a mathematical definition of the Coulomb branch as an affine algebraic variety with $\mathbb{C}^\times$-action when $\mathbf{M}$ is of a form $\mathbf{N} \oplus \mathbf{N}^*$, as the second step of the proposal given in [Nak16].

1. Introduction

This paper is a sequel to [Nak16]. It is written for mathematicians so that no knowledge on physics is required to read. It can be read even independently of [Nak16] except the next background subsection. We by no means intend to ignore past research in physics. We try our best to give appropriate physics references either here or in [Nak16] so that the reader could see how we understand them. There are two companion papers [Quiver, Affine] where specific topics arising from this paper are discussed.

1(i). Background

In [Nak16] the third named author proposed an approach to define the Coulomb branch $\mathcal{M}_C$ of a 3-dimensional $\mathcal{N} = 4$ SUSY gauge theory in a mathematically rigorous way. The Coulomb branch $\mathcal{M}_C$ is a hyper-Kähler manifold with an SU(2)-action possibly with singularities. It has been studied by physicists intensively over the years, nevertheless lacks a firm mathematical footing as the physical definition involves quantum corrections.

A key idea in [Nak16] was to use a certain moduli stack, motivated by reduction of the generalized Seiberg-Witten type equation to $S^2 = \mathbb{P}^1$, associated with a compact Lie group G_c and its representation $\mathbf{M}$ over the quaternions $\mathbb{H}$, corresponding to a given SUSY gauge theory. Then the space of holomorphic functions on $\mathcal{M}_C$ is proposed as the dual of the critical cohomology with compact support of the moduli stack, associated with an analog of the complex Chern-Simons functional.

The goal of this paper is to define a commutative *multiplication* on the dual cohomology group, under the assumption that $\mathbf{M}$ is of cotangent type, i.e., $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ for a complex representation $\mathbf{N}$ of G. The commutative ring of functions determines $\mathcal{M}_C$ as an affine algebraic variety as its spectrum. Therefore the remaining step is to find a hyper-Kähler metric, or equivalently the twistor space.

Under the cotangent type assumption, it was heuristically shown ([Nak16]) that the critical cohomology group can be replaced by the ordinary cohomology group with compact support for a *smaller* moduli stack $\mathcal{R}$ of pairs of holomorphic principal G-bundles $\mathcal{P}$ with holomorphic sections of the associated vector bundle $\mathcal{P} \times_G \mathbf{N}$ over $\mathbb{P}^1$. Here G is a complexification of G_c. In fact, we further change the moduli stack of pairs over $\mathbb{P}^1$ by a stack modeled after the affine Grassmannian. In other words, we consider the moduli stack of pairs over the non-separated scheme $\tilde{D}$ obtained by gluing two copies of the formal disk $D = \operatorname{Spec} \mathbb{C}[[z]]$, along the punctured

disk D^*. This allows us to apply various techniques used in study of affine Grassmannian. (See below for some detail, and the main body for full detail.) The graded dimension of the cohomology group does not change as we will see later, and we conjecture that two cohomology groups are naturally isomorphic.

The original intuition for the multiplication in [Nak16] was a conjectural $3d$ topological quantum field theory associated with the gauge theory. Namely the dual cohomology group is regarded as the quantum Hilbert space $\mathcal{H}_{S^2}$ associated with S^2, and the multiplication is given by the vector corresponding to the 3-ball with two balls removed from the interior. Intuitively our definition of the product uses the same picture, but one of three boundary components is very small. Let us view it as $\Sigma \times [0, 1]$ with small 3-ball B^3 removed from the interior, where $\Sigma = S^2$. Then it can be regarded as a movie of Σ on which a small 2-ball appears and disappears, where $[0, 1]$ is the time direction. (See Figure 1.) We have a multiplication $\mathcal{H}_\Sigma \otimes \mathcal{H}_{\partial B^3} \to \mathcal{H}_\Sigma$ from the $3d$ point of view. On the other hand, from a $2d$ observer on Σ, the intersection $B^3 \cap (\Sigma \times t)$ appears in a small neighborhood of a point. Therefore the observer sees something happening on the formal disk D. From this point of view, $\mathcal{H}_\Sigma$ is a module of a ring $\mathcal{H}_{\partial B^3}$, where the multiplication on $\mathcal{H}_{\partial B^3}$ is given by considering two formal disks. This $2d$ movie is nothing but the convolution diagram for the affine Grassmannian, or more precisely Beilinson-Drinfeld Grassmannian.
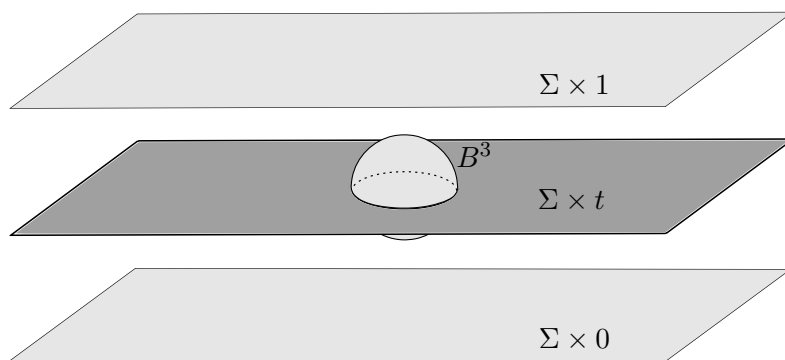

Figure 1: $3d$ picture vs $2d$ movie.

We do not expect our technique to be applied to more general 3-manifolds with boundaries, but we think that it is a good starting point nevertheless.

Let us also mention that the relevance of the affine Grassmannian (and [BFM05], see below) in $3d$ SUSY gauge theories with $\mathbf{N} = 0$ was pointed earlier by Teleman [Tel14].

1(ii). Construction

We believe that our definition of the affine algebraic variety $\mathcal{M}_C$ is interesting in its own, even ignoring physical motivation. For mathematicians who do not know physics, it can be regarded as new construction of a class of algebraic varieties with interesting structures. The class contains moduli spaces of based maps from $\mathbb{P}^1$ to a flag variety (also conjecturally instantons on multi Taub-NUT spaces), and spaces studied in the context of geometric representation theory, such as slices in affine Grassmannian. But we expect $\mathcal{M}_C$ is an unknown space in most cases.

Let us give a little more detail on the definition. Let G be a complex reductive group, and $\mathbf{N}$ be its representation. We consider the moduli space $\mathcal{R}$ of triples $(\mathcal{P}, \varphi, s)$, where $\mathcal{P}$ is a G-bundle on the formal disk $D = \operatorname{Spec} \mathbb{C}[[z]]$, φ is its trivialization over the punctured disk $D^* = \operatorname{Spec} \mathbb{C}((z))$ and s is a section of the associated vector bundle $\mathcal{P} \times_G \mathbf{N}$ such that it is sent to a regular section of a trivial bundle under φ. We have a natural action of $G_{\mathcal{O}} = G[[z]]$, the group of $\mathcal{O}$-valued points of G by changing φ. If we ignore s, we get the moduli space of pairs $(\mathcal{P}, \varphi)$, which is nothing but the affine Grassmannian Gr_G. When $\mathbf{N} = \mathfrak{g}$, the adjoint representation, $\mathcal{R}$ is the affine Grassmannian Steinberg variety. For general $\mathbf{N}$, we have a projection $\mathcal{R} \to \operatorname{Gr}_G$ whose fibers are infinite dimensional vector spaces.

It is a direct limit of inverse limits of schemes of finite type, but its $G_{\mathcal{O}}$-equivariant Borel-Moore homology group $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is well-defined, as we will explain in the main text.

We then introduce a convolution diagram for $\mathcal{R}$ as an analog of one for Gr_G, used in [MV07] for geometric Satake correspondence. It defines a convolution product $*$ on $H_*^{G_{\mathcal{O}}}(\mathcal{R})$. This construction has originally appeared in [BFM05] for Gr_G itself (i.e., $\mathbf{N} = 0$) and $\mathbf{N} = \mathfrak{g}$. There is a closely related earlier work [Vas05] for $\mathbf{N} = \mathfrak{g}$, and also [VV10]. We thus get a graded commutative ring $\mathcal{A}$, and define the Coulomb branch as its spectrum $\mathcal{M}_C = \operatorname{Spec} \mathcal{A}$.

We can thus regard these earlier results as computation of examples of $\mathcal{M}_C$: For $\mathbf{N} = 0$, $\mathcal{M}_C$ is the the algebraic variety $\mathfrak{Z}_{\mathfrak{g}^\vee}^{G^\vee}$ formed by the pairs (g, x) such that x lies in a (fixed) Kostant slice in $\mathfrak{g}^\vee$, and $g \in G^\vee$ satisfies $\operatorname{Ad}_g(x) = x$. See §3(x)(a) for more detail.

For $\mathbf{N} = \mathfrak{g}$, the adjoint representation, $\mathcal{M}_C$ is $(\mathfrak{t} \times T^\vee)/W$, where $\mathfrak{t}$ is the Lie algebra of a maximal torus T, $T^\vee$ is the dual torus of T, and W is the Weyl group. See §3(x)(b) for more detail.

1(iii). Quantization of $\mathcal{M}_C$

As a byproduct of our definition, we obtain a noncommutative deformation (*quantized Coulomb branch*) $\mathcal{A}_\hbar$, flat over $\mathbb{C}[\hbar]$. This is defined as the equivariant homology group $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R})$, where $\mathbb{C}^\times$ is the group of loop rotations, as in earlier works [Vas05, VV10, BF08]. Therefore the classical limit $\mathcal{A}$ has a Poisson bracket. We show that it is symplectic on the regular locus of $\mathcal{M}_C$ (Proposition 6.15). When $\mathbf{N} = 0$, $\mathcal{A}_\hbar$ is a quantum hamiltonian reduction of the ring of differential operators on $G^\vee$ ([BF08, Th. 3]). When $\mathbf{N} = \mathfrak{g}$, $\mathcal{A}_\hbar$ is expected to be the spherical subalgebra of the graded Cherednik algebra, based on [Vas05, VV10]. (See §3(x)(b) for a precise statement.)

The quantized Coulomb branch $\mathcal{A}_\hbar$ contains $H_{G \times \mathbb{C}^\times}^*(\mathrm{pt})1$ as a commutative subalgebra (*Cartan subalgebra*). Correspondingly we have a morphism $\varpi \colon \mathcal{M}_C \to \mathfrak{t}/W \cong \mathbb{C}^\ell$ such that all functions factoring through ϖ are Poisson commuting. A generic fiber of ϖ is $T^\vee$ (Propositions 3.15, 5.19). Thus ϖ is an *integrable system* in the sense of Liouville. In examples $\mathbf{N} = 0$ and $\mathfrak{g}$, the morphism ϖ is an obvious projection to $\mathfrak{t}/W$.

Remarks 1.1. (1) A physical explanation of the 'quantization' is given in [BDG15, §3.4]. It is coming from the Ω-background, i.e., the $\mathbb{C}^\times$-action on $\mathbb{R}^3$. This should have the same origin with our definition.

(2) In 4-dimension, it is known that Coulomb branches are closely related to the Hitchin system (see e.g., [Don97], a review for mathematicians). But the integrable system above is not a simple limit of the 4-dimensional integrable system even for $(G, \mathbf{N}) = (\mathrm{SL}(2), 0)$: $\mathcal{M}_C$ is the Atiyah-Hitchin space, which is the complement of the infinity section in 4d Coulomb branch for $\mathbb{R}^3 \times S^1$, the total space of Seiberg-Witten curves $y^2 = x^3 - x^2 u + x$ ([SW97, (3.18)]), but the hamiltonian is u in 4d while it is $v \overset{\text{def.}}{=} x - u$ in 3d. Anyhow the integrable system is coming from monopole operators corresponding to the cocharacter $\lambda = 0$. And it is proposed that they form a Poisson commuting subalgebra in [BDG15, §3.2, §4.2].

1(iv). The organization of the paper

§2 is devoted to the definition of $\mathcal{R}$ and its equivariant Borel-Moore homology group. We give the definition of the convolution product in §3. Some basic properties of $\mathcal{A}$, except the commutativity of the multiplication, are established also in §3. In §4 we determine $\mathcal{A}$ when G is a torus. We give a linear basis with explicit structure constants. §5 is a technical heart of the paper. We analyze $\mathcal{M}_C$ using the localization theorem in equivariant

homology groups. The compatibility of convolution products with the local-ization is studied. One of proofs of the commutativity of the multiplication is given. (Another proof will be given in [Affine].) We also give a recipe to identify $\mathcal{M}_C$ with a known space, following [BFM05]: Suppose we have $\Pi\colon \mathcal{M} \to \mathfrak{t}/W$ such that $\mathcal{M}$ is normal and all the fibers of Π have the same dimension. If $(\mathcal{M}, \Pi)$ and $(\mathcal{M}_C, \varpi)$ coincide up to codimension 2, they are isomorphic. (See Theorem 5.26 for the precise statement.) In §6 we introduce a degeneration of $\mathcal{M}_C$ to a variety with combinatorial flavor, and prove that $\mathcal{A}$ is finitely generated and normal as applications. We also determine $\mathcal{M}_C$ when G is PGL(2) or SL(2).

There is one appendix. In §A we compare the answer for $\mathbf{N} = 0$ in [BF08] and one in the physics literature for type A. Both are understood in a uniform way that $\mathcal{M}_C$ is the moduli space of solutions of Nahm's equations for the Langlands dual group $G_c^\vee$.

1(v). Companion papers

In [Quiver] we study Coulomb branches of quiver gauge theories. When $(G, \mathbf{N})$ is coming from a quiver of type ADE, $\mathcal{M}_C$ is the moduli space of based maps from $\mathbb{P}^1$ to the flag variety of the corresponding type ADE ([Quiver, Theorem 3.1]). More generally, if $\mathbf{N}$ is coming from a framed quiver of type ADE, $\mathcal{M}_C$ is a slice in the affine Grassmannian of the corresponding type ADE under a dominance condition, and its generalization in general. (See [Quiver, §3].)

In [Affine] we study the following object: As a byproduct of our con-struction we get a $G_{\mathcal{O}}$-equivariant constructible complex $\mathcal{A}$ on Gr_G de-fined by $\pi_* \omega_{\mathcal{R}}[-2\dim \mathbf{N}_{\mathcal{O}}]$, where $\omega_{\mathcal{R}}$ is the dualizing complex on $\mathcal{R}$ and $\pi\colon \mathcal{R} \to \mathrm{Gr}_G$ is the projection. We can recover $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ as $H_{G_{\mathcal{O}}}^*(\mathrm{Gr}_G, \mathcal{A})$. Moreover the construction of the convolution product gives us a homomor-phism $\mathsf{m}\colon \mathcal{A} \star \mathcal{A} \to \mathcal{A}$, where $\star$ is the convolution product on $D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$, the $G_{\mathcal{O}}$-equivariant derived category on Gr_G. Therefore $(\mathcal{A}, \mathsf{m})$ is a ring object in $D_{G_{\mathcal{O}}}(\mathrm{Gr}_G)$.

Notation

1) We basically follow the notation in Part I [Nak16]. However we mainly use a complex reductive group instead of its maximal compact subgroup. Therefore we denote a reductive group by G, and its maximal compact by G_c. On the other hand, we use the notation $\mathcal{R}$ for the variety of triples, though it was used for the corresponding space associated with $\mathbb{P}^1$ in Part I.

2) Let us choose and fix a maximal torus T of G. Let W be the Weyl group of G. Let Y denote the coweight lattice of G. The Lie algebra of G (resp. T) is denoted by $\mathfrak{g}$ (resp. $\mathfrak{t}$).

3) The constant sheaf on a space X is denoted by $\mathbb{C}_X$. The dualizing complex is denoted by $\boldsymbol{\omega}_X$. The Verdier duality is denoted by $\mathbb{D}$. (We take $\mathbb{C}$ as the base ring.)

4) We will not use the usual homology group, and denote the Borel-Moore homology group with complex coefficients by $H_i(X)$. It is $H^{-i}(\boldsymbol{\omega}_X)$, and the dual of cohomology group with compact support. When a group G acts on X, the equivariant Borel-Moore homology group is denoted by $H_i^G(X)$.

Acknowledgments

2. A variety of triples and its equivariant Borel-Moore homology group

Our construction of a space $\mathcal{R}$, a direct limit of inverse limits of schemes of finite type, and its equivariant Borel-Moore homology group are based on similar well-known results for the affine Grassmannian and the affine

Grassmannian Steinberg variety. See [BeiDr, §4.5], [BFM05, §7] and the references therein for the detail and proofs.

2(i). A variety of triples

Let G be a complex connected reductive group. The connectedness assumption is not essential. See Remark 2.8(3) below. Let $\mathcal{O}$ denote the formal power series ring $\mathbb{C}[[z]]$ and $\mathcal{K}$ its fraction field $\mathbb{C}((z))$. Let $\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$ be the affine Grassmannian, where $G_{\mathcal{K}}$, $G_{\mathcal{O}}$ are groups of $\mathcal{K}$ and $\mathcal{O}$-valued points of G. It is an ind-scheme representing the functor from affine schemes to sets:

$$S \mapsto \{(\mathcal{P}, \varphi) \mid \text{a } G\text{-bundle } \mathcal{P} \text{ on } D \times S,$$
$$\text{a trivialization } \varphi \colon \mathcal{P}|_{D^* \times S} \to G \times D^* \times S\},$$

where $D = \mathrm{Spec}(\mathcal{O})$ (resp. $D^* = \mathrm{Spec}(\mathcal{K})$) is the formal disk (resp. punctured disk). We simply say Gr_G is the moduli space of pairs $(\mathcal{P}, \varphi)$ of a G-bundle $\mathcal{P}$ on D and its trivialization φ over D^*. The same applies to the variety of triples, introduced below.

The set $\pi_0(\mathrm{Gr}_G)$ of connected components of Gr_G is known to be in bijection to the fundamental group $\pi_1(G)$ of G. It is a classical result that $\pi_1(G)$ is isomorphic to the quotient of the coweight lattice by the coroot lattice.

Let $\mathbf{N}$ be a (complex) representation of G. We consider a *variety of triples* $\mathcal{R} \equiv \mathcal{R}_{G,\mathbf{N}}$, the moduli space parametrizing triples $(\mathcal{P}, \varphi, s)$, where $(\mathcal{P}, \varphi)$ is in Gr_G, and s is a section of an associated vector bundle $\mathcal{P}_{\mathbf{N}} = \mathcal{P} \times_G \mathbf{N}$ such that it is sent to a regular section of a trivial bundle under φ. We use the notation $\mathcal{R}$ when $(G, \mathbf{N})$ is clear from the context. It is an ind-scheme of ind-infinite type, as we will explain below. But we simply call it a variety.

Let us explain the last condition. Let $\varphi_{\mathbf{N}}$ denote the induced isomorphism $\mathcal{P}_{\mathbf{N}}|_{D^*} \to D^* \times \mathbf{N}$ of vector bundles over D^*. It sends a section $s \in H^0(\mathcal{P}_{\mathbf{N}})$ to a rational section of the trivial bundle $D \times \mathbf{N}$, i.e., an element in $\mathbf{N}_{\mathcal{K}}$. It may have a pole at the origin, as φ is not regular there in general. The last condition means that it is regular, i.e., $\varphi_{\mathbf{N}}(s) \in \mathbf{N}_{\mathcal{O}}$. It defines a finite codimensional subspace in $H^0(\mathcal{P}_{\mathbf{N}})$.

The variety $\mathcal{R}$ is a closed subvariety of a variety $\mathcal{T} \equiv \mathcal{T}_{G,\mathbf{N}}$, the moduli space of $(\mathcal{P}, \varphi, s)$ as above, but s is merely a section of $\mathcal{P}_{\mathbf{N}}$, no further condition on the behavior under $\varphi_{\mathbf{N}}$. Under the projection $\mathcal{T} \to \mathrm{Gr}_G$, it has a structure of a vector bundle over Gr_G, whose fiber at $(\mathcal{P}, \varphi)$ is $H^0(\mathcal{P}_{\mathbf{N}})$.

The rank of the vector bundle is infinite. This is the reason why $\mathcal{T}$ is *not* an ind-scheme of ind-finite type.

The closed embedding $\mathcal{R} \to \mathcal{T}$ is denoted by i. The projection $\mathcal{T} \to \mathrm{Gr}_G$ is denoted by π.

As sets, $\mathcal{T}$ is the quotient $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathbf{N}_{\mathcal{O}}$. Let us write its point as $[g, s]$ with $g \in G_{\mathcal{K}}$, $s \in \mathbf{N}_{\mathcal{O}}$. The equivalence relation is given by $(g, s) \sim (gb^{-1}, bs)$ for $b \in G_{\mathcal{O}}$, and $[g, s]$ denote the equivalence class for (g, s). We have a map $\mathcal{T} \ni [g, s] \mapsto gs \in \mathbf{N}_{\mathcal{K}}$. In terms of the description of $\mathcal{T}$ as a moduli space, gs is nothing but $\varphi_{\mathbf{N}}(s)$. Let us denote it by Π. Together with the projection $\pi \colon \mathcal{T} \to \mathrm{Gr}_G$, it gives a closed embedding $(\pi, \Pi) \colon \mathcal{T} \hookrightarrow \mathrm{Gr}_G \times \mathbf{N}_{\mathcal{K}}$. We have $\mathcal{R} = \mathcal{T} \cap (\mathrm{Gr}_G \times \mathbf{N}_{\mathcal{O}})$.

We have an action of $G_{\mathcal{K}}$ on $\mathcal{T}$ given by the left multiplication. As a moduli space, it is given by the change of the trivialization φ. Its restriction to $G_{\mathcal{O}}$ preserves $\mathcal{R}$. There is also $\mathbb{C}^{\times}$-actions on Gr_G, $\mathcal{T}$, $\mathcal{R}$ induced from the loop rotation of D. It is combined to actions of the semi-direct product $G_{\mathcal{O}} \rtimes \mathbb{C}^{\times}$.

Convention. In order to make formulas look cleaner, we let the loop rotation $\mathbb{C}^{\times}$ act on $\mathbf{N}$ by weight $1/2$. (So we take a double covering of $\mathbb{C}^{\times}$, but we do not introduce a new notation for the double cover for brevity.)

Remark 2.1. Let St be the usual (finite dimensional) Steinberg variety, i.e.,

$$\mathrm{St} = \{(B_1, x, B_2) \in \mathfrak{B} \times \mathfrak{g} \times \mathfrak{B} \mid x \in \mathfrak{n}_1 \cap \mathfrak{n}_2\},$$

where $\mathfrak{B}$ is the flag variety, considered as the space of Borel subgroups, and $\mathfrak{n}_a$ is the nilradical of the Lie algebra of B_a ($a = 1, 2$). We have an action of $G \times \mathbb{C}^{\times}$ on St, and the equivariant Borel-Moore homology group $H_*^{G \times \mathbb{C}^{\times}}(\mathrm{St})$ gives a geometric realization of the degenerate affine Hecke algebra [Lus88] (see also [CG97] for the K-theory version).

If we fix a point $B_2 = B \in \mathfrak{B}$, we have an isomorphism $\mathfrak{B} = G/B$, and the induced isomorphism $\mathrm{St} \cong G \times_B \overline{\mathrm{St}}$, where

$$\overline{\mathrm{St}} = \{(B_1, x) \in \mathfrak{B} \times \mathfrak{g} \mid x \in \mathfrak{n}_1 \cap \mathfrak{n}\},$$

where $\mathfrak{n}$ is the nilradical of the Lie algebra of B. Our space $\mathcal{R}$ is an analogue of $\overline{\mathrm{St}}$. We have

$$H_*^{G \times \mathbb{C}^{\times}}(\mathrm{St}) \cong H_*^{B \times \mathbb{C}^{\times}}(\overline{\mathrm{St}}),$$

hence we can understand the geometric realization in terms of $\overline{\mathrm{St}}$.

Let Y be the coweight lattice of G. It is well-known that $G_{\mathcal{O}}$-orbits in Gr_G are parametrized by dominant coweights Y^+: $\mathrm{Gr}_G = \bigsqcup_{\lambda \in Y^+} \mathrm{Gr}_G^\lambda$. The closure relation corresponds to the usual order on Y^+: $\overline{\mathrm{Gr}}_G^\lambda = \bigsqcup_{\mu \leq \lambda} \mathrm{Gr}_G^\mu$. It is well-known that $\overline{\mathrm{Gr}}_G^\lambda$ is a scheme of finite type. The $G_{\mathcal{O}}$-action on $\overline{\mathrm{Gr}}_G^\lambda$ factors through a finite dimensional quotient.

Recall $\pi \colon \mathcal{T} \to \mathrm{Gr}_G$ denote the projection forgetting sections. Let $\mathcal{T}_{\leq \lambda} \overset{\text{def.}}{=} \pi^{-1}(\overline{\mathrm{Gr}}_G^\lambda)$. This is a scheme of infinite type, and $\mathcal{T} = \bigcup \mathcal{T}_{\leq \lambda}$. We define $\mathcal{R}_{\leq \lambda} \overset{\text{def.}}{=} \mathcal{R} \cap \pi^{-1}(\overline{\mathrm{Gr}}_G^\lambda)$. It is also a scheme of infinite type and $\mathcal{R} = \bigcup \mathcal{R}_{\leq \lambda}$.

For a sufficiently large $d \gg 0$ consider the fiberwise translation by $z^d \mathbf{N}_{\mathcal{O}}$ on $\mathcal{T} = G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathbf{N}_{\mathcal{O}}$. The quotient is $\mathcal{T} = G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathbf{N}_{\mathcal{O}} \to \mathcal{T}^d \overset{\text{def.}}{=} G_{\mathcal{K}} \times_{G_{\mathcal{O}}} (\mathbf{N}_{\mathcal{O}}/z^d \mathbf{N}_{\mathcal{O}})$. We have a surjective vector bundle homomorphism $p_e^d \colon \mathcal{T}^d \to \mathcal{T}^e$ for $d > e$. The original $\mathcal{T}$ can be understood as the inverse limit of this system. Let $\mathcal{T}_{\leq \lambda}^d = (\pi^d)^{-1}(\overline{\mathrm{Gr}}_G^\lambda)$ where $\pi^d \colon \mathcal{T}^d \to \mathrm{Gr}_G$ is the projection. It is a scheme of finite type. Moreover we have an induced homomorphism $\mathcal{T}_{\leq \lambda}^d \to \mathcal{T}_{\leq \lambda}^e$, and $\mathcal{T}_{\leq \lambda}$ is the inverse limit of this system.

The order of pole of φ at 0 is bounded by a constant depending on λ for $(\mathcal{P}, \varphi) \in \overline{\mathrm{Gr}}_G^\lambda$. Therefore $\mathcal{R}_{\leq \lambda}$ is invariant under the translation by $z^d \mathbf{N}_{\mathcal{O}}$ if we choose d larger than the order. Let $\mathcal{R}_{\leq \lambda}^d$ denote the quotient. It is a closed subscheme of $\mathcal{T}_{\leq \lambda}^d$. Moreover we have an affine fibration $\tilde{p}_e^d \colon \mathcal{R}_{\leq \lambda}^d \to \mathcal{R}_{\leq \lambda}^e$, as the restriction of p_e^d for $d > e$. Then $\mathcal{R}_{\leq \lambda}$ is the inverse limit of this system.

Let $\mathcal{R}_\lambda = \mathcal{R} \cap \pi^{-1}(\mathrm{Gr}_G^\lambda)$, the inverse image of the $G_{\mathcal{O}}$-orbit Gr_G^λ. It is an open subvariety in a closed subvariety $\mathcal{R}_{\leq \lambda}$, hence locally closed in $\mathcal{R}$. Let $\mathcal{R}_{<\lambda}$ be the complement $\mathcal{R}_{\leq \lambda} \setminus \mathcal{R}_\lambda$. It is closed subvariety. Let us define $\mathcal{T}_\lambda$, $\mathcal{T}_{<\lambda}$ in the same way.

Lemma 2.2. *The restriction of π to $\mathcal{R}_\lambda$ is a vector bundle $\mathcal{R}_\lambda \to \mathrm{Gr}_G^\lambda$ of infinite rank. It is a subbundle of another infinite rank vector bundle $\mathcal{T}_\lambda \to \mathrm{Gr}_G^\lambda$ such that the quotient bundle has a finite rank given by the formula*

$$d_\lambda \overset{\text{def.}}{=} \mathrm{rank}(\mathcal{T}_\lambda/\mathcal{R}_\lambda) = \sum_\chi \max(-\langle \chi, \lambda \rangle, 0) \dim \mathbf{N}(\chi),$$

where $\mathbf{N}(\chi)$ is the weight χ subspace of $\mathbf{N}$.

Proof. This is obvious since $\mathcal{R}_\lambda$ is $G_{\mathcal{O}}$-invariant and Gr_G^λ is a $G_{\mathcal{O}}$-orbit: Consider a coweight z^λ as an element $G_{\mathcal{K}}$, and also a point in Gr_G^λ. Then the fiber $\mathcal{R} \cap \pi^{-1}(\lambda)$ is $\mathbf{N}_{\mathcal{O}} \cap z^\lambda \mathbf{N}_{\mathcal{O}} = \{s \in \mathbf{N}_{\mathcal{O}} \mid z^{-\lambda} s \in \mathbf{N}_{\mathcal{O}}\}$. This is a subspace of $\mathbf{N}_{\mathcal{O}}$ invariant under the stabilizer $\mathrm{Stab}_{G_{\mathcal{O}}}(z^\lambda)$. Then $\mathcal{R}_\lambda$ is the vector bundle over $\mathrm{Gr}_G^\lambda = G_{\mathcal{O}}/\mathrm{Stab}_{G_{\mathcal{O}}}(z^\lambda)$ associated with $\mathbf{N}_{\mathcal{O}} \cap z^\lambda \mathbf{N}_{\mathcal{O}}$.

The rank of the quotient is the dimension of $z^\lambda \mathbf{N}_\mathcal{O} / \mathbf{N}_\mathcal{O} \cap z^\lambda \mathbf{N}_\mathcal{O}$. This can be computed by decomposing $\mathbf{N}$ into weight spaces. If s is contained in the weight χ subspace, we replace $\mathbf{N}$ by a 1-dimensional subspace, and find that the contribution is $\dim z^{\langle \chi, \lambda \rangle} \mathbb{C}[[z]] / \mathbb{C}[[z]] \cap z^{\langle \chi, \lambda \rangle} \mathbb{C}[[z]]$. This is equal to $\max(-\langle \chi, \lambda \rangle, 0)$. The above formula follows. $\square$

2(ii). Equivariant Borel-Moore homology of the variety of triples

We shall use the equivariant Borel-Moore homology groups $H_*^{G_\mathcal{O}}(\mathcal{R})$ and $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R})$ to define the Coulomb branch and its quantization. Since both $G_\mathcal{O}$ and $\mathcal{R}$ are infinite dimensional, these homology groups must be treated carefully.

We have the ind-scheme $\mathcal{R}$ together with a map $\pi \colon \mathcal{R} \to \mathrm{Gr}_G$. We would like to define its $G_\mathcal{O}$-equivariant Borel-Moore homology. To this end it is sufficient to define the $G_\mathcal{O}$-equivariant Borel-Moore homology of $\mathcal{R}_{\leq \lambda} = \mathcal{R} \cap \pi^{-1}(\overline{\mathrm{Gr}}_G^\lambda)$ in such a way that an embedding $\mathcal{R}_{\leq \mu} \hookrightarrow \mathcal{R}_{\leq \lambda}$ will induce a map $H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq \mu}) \to H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq \lambda})$ for $\mu \leq \lambda$.

Now, given such $\mathcal{R}_{\leq \lambda}$, we choose an integer $d \geq 0$ so that we have a finite dimensional scheme $\mathcal{R}_{\leq \lambda}^d$ as its quotient as above. We have the induced $G_\mathcal{O}$-action on $\mathcal{R}_{\leq \lambda}^d$. Let $G_i = G(\mathcal{O}/z^i \mathcal{O})$. This is a quotient of $G_\mathcal{O}$ and for large i the action of $G_\mathcal{O}$ factorizes through G_i.

We now set

$$H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq \lambda}) \overset{\text{def.}}{=} H_{G_i}^{-*}(\mathcal{R}_{\leq \lambda}^d, \boldsymbol{\omega}_{\mathcal{R}_{\leq \lambda}^d})[-2 \dim(\mathbf{N}_\mathcal{O}/z^d \mathbf{N}_\mathcal{O})].$$

We claim that this definition depends neither on i nor on d. Indeed, independence of i follows from the fact that for $i > j$ we have a surjective map $G_i \to G_j$ with unipotent kernel. Independence of d follows from the fact that for $d > e$ we have a $G_\mathcal{O}$-equivariant map $\tilde{p}_e^d \colon \mathcal{R}_{\leq \lambda}^d \to \mathcal{R}_{\leq \lambda}^e$ which is a locally trivial fibration with fibers being affine spaces of dimension $\dim(z^e \mathbf{N}_\mathcal{O}/z^d \mathbf{N}_\mathcal{O})$. Note that if $p \colon Z \to W$ is a locally trivial fibration of finite-dimensional schemes over $\mathbb{C}$ with fibers being affine spaces of dimension r then we have a canonical isomorphism $H^*(Z, \boldsymbol{\omega}_Z) \simeq H^*(W, \boldsymbol{\omega}_W)[2r]$ and the same is true for equivariant Borel-Moore homology with respect to any algebraic group K acting on Z and W (and such that the morphism p is K-equivariant).

Note also that the degree of this homology group is given relative to '$2 \dim \mathbf{N}_\mathcal{O}$'. Namely if a homology classes has degree k, it means that we consider homology classes $\mathcal{R}_{\leq \lambda}^d$ for all sufficiently large d whose degree is $k + 2 \dim(\mathbf{N}_\mathcal{O}/z^d \mathbf{N}_\mathcal{O})$. As $d \to \infty$, the degree goes to '$k + 2 \dim \mathbf{N}_\mathcal{O}$'. Since

it is not illuminating to go back to finite dimensional approximations every time, we use this convention hereafter: we just write $2 \dim \mathbf{N}_{\mathcal{O}}$ (and later $2 \dim G_{\mathcal{O}}$) without mentioning finite dimensional approximations.

Given an embedding $\mathcal{R}_{\leq\mu} \hookrightarrow \mathcal{R}_{\leq\lambda}$ as above, note that for sufficiently large d and i we have a G_i-equivariant closed embedding $\mathcal{R}_{\leq\mu}^d \hookrightarrow \mathcal{R}_{\leq\lambda}^d$ and we can use the push-forward with respect to this closed embedding to define the map

$$H_{G_i}^*(\mathcal{R}_{\leq\mu}^d, \boldsymbol{\omega}_{\mathcal{R}_{\leq\mu}^d})[-2\dim(\mathbf{N}_{\mathcal{O}}/z^d\mathbf{N}_{\mathcal{O}})]$$
$$\to H_{G_i}^*(\mathcal{R}_{\leq\lambda}^d, \boldsymbol{\omega}_{\mathcal{R}_{\leq\lambda}^d})[-2\dim(\mathbf{N}_{\mathcal{O}}/z^d\mathbf{N}_{\mathcal{O}})].$$

The equivariant Borel-Moore homology group $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is a module over $H_{G_{\mathcal{O}}}^*(\mathrm{pt})$, the equivariant cohomology group of a point, defined using G_i above. Since any G_i acts (trivially) on pt, we have a natural isomorphism $H_{G_{\mathcal{O}}}^*(\mathrm{pt}) \cong H_G^*(\mathrm{pt})$.

The definition of the $G_{\mathcal{O}} \rtimes \mathbb{C}^\times$-equivariant homology group is the same.

In the definition of the convolution product, we also use equivariant Borel-Moore homology groups of other spaces (see (3.2)). A prototype of such homology groups is $H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}})$. Let us explain how this is defined. Homology groups of spaces actually needed are simple variants of $H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}})$, and hence can be defined in the same way.

Let $G_i = G(\mathcal{O}/z^i\mathcal{O})$ as before. We have a surjective homomorphism $G_{\mathcal{O}} \to G_i$, and let K_i be its kernel. Take a dominant coweight λ, and let $G_{\mathcal{K}}^{\leq\lambda}$ be the inverse image of $\overline{\mathrm{Gr}}_G^\lambda$ under $G_{\mathcal{K}} \to \mathrm{Gr}_G$. We take $j \gg i$ so that K_j acts trivially on $G_{\mathcal{K}}^{\leq\lambda}/K_i$. (This is possible by the same well-known argument that K_i acts trivially on $\overline{\mathrm{Gr}}_G^\lambda$ for $i \gg 0$.) Then we have an action of $G_j \times G_i$ on $G_{\mathcal{K}}^{\leq\lambda}/K_i$. We define

$$H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}}^{\leq\lambda}) \overset{\text{def.}}{=} H_{G_j \times G_i}^{-*}(G_{\mathcal{K}}^{\leq\lambda}/K_i, \boldsymbol{\omega}_{G_{\mathcal{K}}^{\leq\lambda}/K_i})[-2\dim G_i].$$

This definition is independent of i or j by the same argument as above. Note also that the degree is given relative to '$2\dim G_{\mathcal{O}}$' in the same sense as above.

We have a homomorphism $H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}}^{\leq\mu}) \to H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}}^{\leq\lambda})$ for $\mu \leq \lambda$ as above. Therefore we define $H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}})$ as the direct limit of this system.

Furthermore, as $G_{\mathcal{K}}^{\leq\lambda}/K_i \to \overline{\mathrm{Gr}}_G^\lambda$ is a principal G_i-bundle, we have

$$H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}}^{\leq\lambda}) \cong H_*^{G_{\mathcal{O}}}(\overline{\mathrm{Gr}}_G^\lambda).$$

As a direct limit, we have an isomorphism

$$(2.3) \qquad\qquad H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}}) \cong H_*^{G_{\mathcal{O}}}(\mathrm{Gr}_G).$$

Remark 2.4. We consider Borel-Moore homology groups with *complex* coefficients. But many of results below remain true for *integer* coefficients. In particular, our Coulomb branch $\mathcal{M}_C$ will be an affine scheme over the integers. We leave to the interested reader the consideration of possible applications, and stick to complex coefficients in this paper.

2(iii). Monopole formula

We compute the equivariant Poincaré polynomial of $\mathcal{R}$ in this subsection. This computation is essentially the same as [Nak16, §8], but we give the detail as it is simple.

We take the following convention:

$$P_t^{G_{\mathcal{O}}}(\mathcal{R}) = \sum_k t^{-k} \dim H_k^{G_{\mathcal{O}}}(\mathcal{R}),$$

and similarly for other spaces.

Let Gr_G^λ be the $G_{\mathcal{O}}$-orbit corresponding to a dominant coweight λ as before. Let

$$P_G(t; \lambda) \overset{\mathrm{def.}}{=} \prod \frac{1}{1 - t^{2d_i}},$$

where d_i are degrees of generators of invariant polynomials of $\mathrm{Stab}_G(\lambda)$. In other words, they are *exponents* plus one.

Lemma 2.5. *The equivariant Poincaré polynomial of Gr_G^λ is given by the formula*

$$P_t^{G_{\mathcal{O}}}(\mathrm{Gr}_G^\lambda) = t^{-4\langle \rho, \lambda \rangle} P_G(t; \lambda),$$

where ρ is the half sum of positive roots, and λ is taken so that it is dominant.

If λ is dominant, we have

$$-4\langle \rho, \lambda \rangle = -2 \sum_{\alpha \in \Delta^+} |\langle \alpha, \lambda \rangle| = -\sum_{\alpha \in \Delta} |\langle \alpha, \lambda \rangle|,$$

where Δ^+ (resp. Δ) is the set of positive (resp. all) roots. The right hand side is invariant under the Weyl group action, while the left hand side is *not*. Since $P_t^{G_{\mathcal{O}}}(\mathrm{Gr}_G^\lambda)$ is W-invariant, it is better to replace $-4\langle \rho, \lambda \rangle$ by the right hand side.

Proof. It is known that Gr_G^λ is a vector bundle over a flag manifold G/P_λ, where P_λ is the parabolic subgroup associated with λ. It is also known that dimension of Gr_G^λ is $2\langle\rho,\lambda\rangle$. (See e.g., [MV07, §2]. In fact, the tangent space of Gr_G^λ at z^μ is isomorphic to $\bigoplus_{\alpha\in\Delta}\bigoplus_{n=0}^{\max(0,\langle\alpha,\mu\rangle)-1}\mathfrak{g}_\alpha z^n$.)

We have

$$H_*^{G_\mathcal{O}}(\mathrm{Gr}_G^\lambda) = H_*^G(\mathrm{Gr}_G^\lambda) \cong H^G_{*-4\langle\rho,\lambda\rangle+2\dim G/P_\lambda}(G/P_\lambda)$$
$$\cong H_G^{-*+4\langle\rho,\lambda\rangle}(G/P_\lambda) \cong H_{\mathrm{Stab}_G(\lambda)}^{-*+4\langle\rho,\lambda\rangle}(\mathrm{pt}),$$

where $\mathrm{Stab}_G(\lambda)$ is as above, which is the Levi quotient of P_λ. Now the assertion follows from the well-known result $P_{1/t}^{\mathrm{Stab}_G(\lambda)}(\mathrm{pt}) = \prod 1/(1-t^{2d_i})$. □

Lemma 2.6. (1) *The equivariant Poincaré polynomial of $H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda)$ is given by*

$$P_t^{G_\mathcal{O}}(\mathcal{R}_\lambda) = t^{2d_\lambda} P_t^{G_\mathcal{O}}(\mathrm{Gr}_G^\lambda).$$

In particular, homology group vanishes in odd degrees.

(2) *The homology group $H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq\lambda})$ vanishes in odd degrees. Hence the Mayer-Vietoris sequence splits into short exact sequences*

$$0 \to H_*^{G_\mathcal{O}}(\mathcal{R}_{<\lambda}) \to H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq\lambda}) \to H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda) \to 0.$$

Proof. (1) Since $\mathcal{R}_\lambda \to \mathrm{Gr}_G^\lambda$ is a vector bundle (see Lemma 2.2), we have the Gysin isomorphism $H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda) \cong H_{*+2d_\lambda}^{G_\mathcal{O}}(\mathrm{Gr}_G^\lambda)$. Here note that the rank of $\mathcal{R}_\lambda$ is $2\dim\mathbf{N}_\mathcal{O} - 2d_\lambda$, as the rank of $\mathcal{T}$ is $2\dim\mathbf{N}_\mathcal{O}$. Since the degree of $H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda)$ is relative to $2\dim\mathbf{N}_\mathcal{O}$, we have the above shift of the degree. The formula of the equivariant Poincaré polynomial follows. The vanishing of odd degree homology follows from Lemma 2.5 above.

(2) We prove the vanishing of $H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq\lambda})$ by induction on λ. If λ is a minimal element, $\mathcal{R}_{\leq\lambda} = \mathcal{R}_\lambda$, and hence the assertion is true by above. For general λ, we have odd degree vanishing of $H_*^{G_\mathcal{O}}(\mathcal{R}_{<\lambda})$, $H_*^{G_\mathcal{O}}(\mathcal{R}_\lambda)$ by the induction hypothesis and the above. Looking at the Mayer-Vietoris long exact sequence for the triple $(\mathcal{R}_{<\lambda}, \mathcal{R}_{\leq\lambda}, \mathcal{R}_\lambda)$, we have the odd vanishing of $H_*^{G_\mathcal{O}}(\mathcal{R}_{\leq\lambda})$. □

We thus get

Proposition 2.7. *Fix a dominant coweight $\overline{\lambda}$. Then*

$$P_t^{G_{\mathcal{O}}}(\mathcal{R}_{\leq\overline{\lambda}}) = \sum_{\lambda \leq \overline{\lambda}} t^{2d_\lambda - 4\langle\rho,\lambda\rangle} P_G(t; \lambda),$$

where the sum runs over dominant coweights λ with $\lambda \leq \overline{\lambda}$.

Remarks 2.8. (1) Taking $\overline{\lambda} \to \infty$, we formally get

$$(2.9) \qquad P_t^{G_{\mathcal{O}}}(\mathcal{R}) = \sum_{\lambda} t^{2d_\lambda - 4\langle\rho,\lambda\rangle} P_G(t; \lambda).$$

However this infinite sum may not be well-defined even as a formal Laurent series, as we do not have a control on $2d_\lambda - 4\langle\rho, \lambda\rangle$ in general.

(2) The above formal infinite sum is essentially the same as the monopole formula of the Hilbert series of the Coulomb branch of the 3-dimensional $\mathcal{N} = 4$ SUSY gauge theory associated with $(G_c, \mathbf{N} \oplus \mathbf{N}^*)$, proposed by Cremonesi, Hanany and Zaffaroni [CHZ14]. Here there is a slight difference: $d_\lambda - 2\langle\rho, \lambda\rangle$ is replaced by

$$(2.10) \qquad \Delta(\lambda) \stackrel{\text{def.}}{=} -\sum_{\alpha \in \Delta^+} |\langle\alpha, \lambda\rangle| + \frac{1}{2}\sum_{\chi} |\langle\chi, \lambda\rangle| \dim \mathbf{N}(\chi).$$

It is a simple exercise to check that the difference

$$d_\lambda - 2\langle\rho, \lambda\rangle - \Delta(\lambda) = -\frac{1}{2}\sum_{\chi}\langle\chi, \lambda\rangle \dim \mathbf{N}(\chi)$$

depends only on the equivalence class $[\lambda]$ in $\pi_1(G) = \pi_0(\mathcal{R})$. (In fact, it depends only on the free part of the abelian group $\pi_1(G)$.) Therefore this correction term is harmless: we just shift the degree on each component of $\mathcal{R}$. This shift turns out to be natural when we identify the Coulomb branch with known examples. The degree $\Delta(\lambda)$ is determined so that the corresponding S^1-action, the restriction of the $\mathbb{C}^\times$-action in §3(v), extends to an SU(2)-action on the Coulomb branch which rotates the hyperKähler structure.[1] See §4(iv) below. See also [Quiver, Remarks 3.3 and 3.13].

It should be remarked also that the monopole formula is proposed under the assumption $2\Delta(\lambda) \geq 1$ for any $\lambda \neq 0$ (the so-called 'good' or 'ugly' theory.) This is to avoid a possible divergence in the infinite sum above pointed

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[1] The third named author thanks Amihay Hanany for his explanation.

out in (1). On the other hand, our $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is always well-defined even without this assumption. We do not have any problem even if its degree piece is infinite dimensional.

Nevertheless the monopole formula is very useful to investigate expected properties of the Coulomb branch.

(3) Suppose that G is possibly disconnected. Since Gr_G depends on the connected component G^0 of G, our variety $\mathcal{R}$ of triples does not see the component group. However the equivariant homology group *does* see the component group: $H_*^{G_{\mathcal{O}}}(\mathcal{R}) \cong H_*^{G_{\mathcal{O}}^0}(\mathcal{R})^\Gamma$, where $\Gamma = G/G^0$. (See e.g., [Hsi75, Chap. 3, §1, Example 3].) For the Coulomb branch defined in the next section, it means $\mathcal{M}_C(G, \mathbf{N}) = \mathcal{M}_C(G^0, \mathbf{N})/\Gamma$. For the monopole formula above, we should understand $P_G(t; \lambda)$ as the Poincaré polynomial of $H_{\mathrm{Stab}_G(\lambda)}^*(\mathrm{pt})$, where $\mathrm{Stab}_G(\lambda)$ is possibly disconnected. See [CHMZ15, App. A] for examples of computation.

3. Definition of Coulomb branches as affine schemes

We define the convolution product on $H_*^{G_{\mathcal{O}}}(\mathcal{R})$, following [BFM05, §7] in this section. This gives us a definition of the Coulomb branch as the spectrum of $H_*^{G_{\mathcal{O}}}(\mathcal{R})$.

We use a sheaf theoretic framework for later applications, hence need to make some points in the construction [BFM05, §7] to actual statements, e.g., Lemma 3.5.

3(i). Convolution diagram

Recall the convolution diagram for the affine Grassmannian ([MV07, (4.1)]):

$$(3.1) \qquad \mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{\quad p \quad} G_{\mathcal{K}} \times \mathrm{Gr}_G \xrightarrow{\quad q \quad} \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \xrightarrow{\quad m \quad} \mathrm{Gr}_G,$$

Here $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ is the quotient $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathrm{Gr}_G$. The maps p, q are projections and m is the multiplication. For $G_{\mathcal{O}}$-equivariant perverse sheaves A_1, A_2, the pullback $p^*(A_1 \boxtimes A_2)$ descends to $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ by the equivariance. Let us denote it by $A_1 \tilde{\boxtimes} A_2$. Then we define $A_1 \star A_2$ by $m_*(A_1 \tilde{\boxtimes} A_2)$. It defines a symmetric monoidal structure on the category of $G_{\mathcal{O}}$-equivariant perverse sheaves on Gr_G, and is equivalent to the monoidal category of finite dimensional representations of the Langlands dual of G [MV07].

Let us describe the convolution diagram (3.1) in terms of functors, as in §2(i). This is given in [MV07, §5]. The leftmost space $\mathrm{Gr}_G \times \mathrm{Gr}_G$ is the moduli space of $(\mathcal{P}_1, \varphi_1, \mathcal{P}_2, \varphi_2)$, two G-bundles on the formal disk D with

trivializations over the punctured disk D^*. The next $G_{\mathcal{K}} \times \mathrm{Gr}_G$ is the moduli of $(\mathcal{P}_1, \varphi_1, \kappa, \mathcal{P}_2, \varphi_2)$, the above data together with a trivialization κ of $\mathcal{P}_1$ over D. The third space $\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G$ is the moduli of $(\mathcal{P}_1, \varphi_1, \mathcal{P}_2, \eta)$, where η is an isomorphism between $\mathcal{P}_1$ and $\mathcal{P}_2$ over D^*.

The map p just forgets κ. The map q is

$$(\mathcal{P}_1, \varphi_1, \kappa, \mathcal{P}_2, \varphi_2) \mapsto (\mathcal{P}_1, \varphi_1, \mathcal{P}_2, \varphi_2^{-1} \circ \kappa|_{D^*}).$$

Note $\varphi_2^{-1} \circ \kappa|_{D^*}$ is an isomorphism from $\mathcal{P}_1|_{D^*}$ to $\mathcal{P}_2|_{D^*}$, as required. Finally m is given by $(\mathcal{P}_1, \varphi_1, \mathcal{P}_2, \eta) \mapsto (\mathcal{P}_2, \varphi_1 \circ \eta^{-1})$.

The goal of this subsection is to introduce corresponding diagrams for $\mathcal{R}$. Recall that $\mathcal{T}$ is the quotient $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathbf{N}_{\mathcal{O}}$, and we have an embedding $\mathcal{T} \hookrightarrow \mathrm{Gr}_G \times \mathbf{N}_{\mathcal{K}}$ such that $\mathcal{R} = \mathcal{T} \cap (\mathrm{Gr}_G \times \mathbf{N}_{\mathcal{O}})$. We consider the induced space $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{R}$. It consists of $[g_1, [g_2, s]]$ with $g_1 \in G_{\mathcal{K}}$, $[g_2, s] \in \mathcal{R} \subset \mathcal{T} = G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathbf{N}_{\mathcal{O}}$. We have $[g_1, [g_2, s]] = [g_1 b, [b^{-1} g_2, s]]$ for $b \in G_{\mathcal{O}}$. We consider the diagram

$$(3.2) \quad \begin{array}{ccccccc}
\mathcal{R} \times \mathcal{R} & \xleftarrow{\tilde{p}} & p^{-1}(\mathcal{R} \times \mathcal{R}) & \xrightarrow{\tilde{q}} & q(p^{-1}(\mathcal{R} \times \mathcal{R})) & \xrightarrow{\tilde{m}} & \mathcal{R} \\
{\scriptstyle i \times \mathrm{id}_{\mathcal{R}}} \downarrow & & {\scriptstyle i'} \downarrow & & \downarrow & & \downarrow {\scriptstyle i} \\
\mathcal{T} \times \mathcal{R} & \xleftarrow{p} & G_{\mathcal{K}} \times \mathcal{R} & \xrightarrow{q} & G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{R} & \xrightarrow{m} & \mathcal{T},
\end{array}$$

where the first row consists of closed subvarieties in spaces in the second row. Maps in the second row are given by

$$(3.3) \qquad ([g_1, g_2 s], [g_2, s]) \leftarrow\!\shortmid (g_1, [g_2, s]) \mapsto [g_1, [g_2, s]] \mapsto [g_1 g_2, s].$$

Since $p^{-1}(\mathcal{R} \times \mathcal{R}) = \{(g_1, [g_2, s]) \mid g_1 g_2 s \in \mathbf{N}_{\mathcal{O}}\}$, the target of $\tilde{m}$ is $\mathcal{R}$ as required. The map m factors as $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{R} \xrightarrow{\mathrm{id}_{G_{\mathcal{K}}} \times_{G_{\mathcal{O}}} i} G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{T} \xrightarrow{m'} \mathcal{T}$, and m' (and hence also m) is ind-proper.

Let us introduce the following group actions on terms in the second row:

$$(3.4) \quad \begin{array}{c}
G_{\mathcal{O}} \times G_{\mathcal{O}} \curvearrowright \mathcal{T} \times \mathcal{R}; \; (g, h) \cdot ([g_1, s_1], [g_2, s_2]) = ([g g_1, s_1], [h g_2, s_2]), \\
G_{\mathcal{O}} \times G_{\mathcal{O}} \curvearrowright G_{\mathcal{K}} \times \mathcal{R}; \; (g, h) \cdot (g_1, [g_2, s]) = (g g_1 h^{-1}, [h g_2, s]), \\
G_{\mathcal{O}} \curvearrowright G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{R}; \; g \cdot [g_1, [g_2, s]] = [g g_1, [g_2, s]], \\
G_{\mathcal{O}} \curvearrowright \mathcal{T}; \; g \cdot [g_1, s] = [g g_1, s].
\end{array}$$

These actions preserve spaces in the first row. Moreover morphisms p, q, m are equivariant, where we take a group homomorphism $p_1 \colon G_{\mathcal{O}} \times G_{\mathcal{O}} \to G_{\mathcal{O}}$ for q.

Let us understand spaces in (3.2) as moduli spaces. Let us first consider spaces in the lower row.

The space $\mathcal{T} \times \mathcal{R}$ is clear. It is the moduli space of $(\mathcal{P}_1, \varphi_1, s_1, \mathcal{P}_2, \varphi_2, s_2)$ such that $(\mathcal{P}_1, \varphi_1)$, $(\mathcal{P}_2, \varphi_2) \in \mathrm{Gr}_G$ and s_1, s_2 are sections of the associated bundles $\mathcal{P}_{1,\mathbf{N}}$, $\mathcal{P}_{2,\mathbf{N}}$. We require $\varphi_{2,\mathbf{N}}(s_2) \in \mathbf{N}_{\mathcal{O}}$. The second space $G_{\mathcal{K}} \times \mathcal{R}$ is the moduli of $(\mathcal{P}_1, \varphi_1, \kappa, \mathcal{P}_2, \varphi_2, s_2)$, where κ is a trivialization of $\mathcal{P}_1$ as in the affine Grassmannian case. The third space $G_{\mathcal{K}} \times_{G_{\mathcal{O}}} \mathcal{R}$ is the moduli of $(\mathcal{P}_1, \varphi_1, \mathcal{P}_2, s_2, \eta)$, where η is an isomorphism between $\mathcal{P}_1$ and $\mathcal{P}_2$ over D^*. We require $\eta_{\mathbf{N}}^{-1}(s_2) \in H^0(\mathcal{P}_{1,\mathbf{N}})$.

Let us describe maps. The map p is given as follows. Let us take $(\mathcal{P}_1, \varphi_1, \kappa, \mathcal{P}_2, \varphi_2, s_2)$ from $G_{\mathcal{K}} \times \mathcal{R}$. Since $\varphi_{2,\mathbf{N}}(s_2) \in \mathbf{N}_{\mathcal{O}}$, it can be considered as a section of the trivial bundle $D \times \mathbf{N}$ over D. We transfer it to a section of $\mathcal{P}_{1,\mathbf{N}}$ by $\kappa_{\mathbf{N}} \colon \mathcal{P}_{1,\mathbf{N}} \to D \times \mathbf{N}$. We denote it by $\kappa_{\mathbf{N}}^{-1} \circ \varphi_{2,\mathbf{N}}(s_2)$. Then we have $(\mathcal{P}_1, \varphi_1, \kappa_{\mathbf{N}}^{-1} \circ \varphi_{2,\mathbf{N}}(s_2), \mathcal{P}_2, \varphi_2, s_2) \in \mathcal{T} \times \mathcal{R}$. The map q is given by

$$(\mathcal{P}_1, \varphi_1, \kappa, \mathcal{P}_2, \varphi_2, s_2) \mapsto (\mathcal{P}_1, \varphi_1, s_2, \mathcal{P}_2, \varphi_2^{-1} \circ \kappa|_{D^*}).$$

The condition $\varphi_{2,\mathbf{N}}(s_2) \in \mathbf{N}_{\mathcal{O}}$ is equivalent to

$$\eta_{\mathbf{N}}^{-1}(s_2) = \kappa_{\mathbf{N}}^{-1} \varphi_{2,\mathbf{N}}(s_2) \in H^0(\mathcal{P}_{1,\mathbf{N}}).$$

Since we do not need to touch the section s_2, it is essentially the same as the corresponding map in the affine Grassmannian case. Finally the map m is given by $(\mathcal{P}_1, \varphi_1, \mathcal{P}_2, s_2, \eta) \mapsto (\mathcal{P}_2, \varphi_1 \circ \eta^{-1}, s_2)$. This is again the same as the affine Grassmannian case, but we note that there is no reason that the trivialization $\varphi_1 \circ \eta^{-1}$ sends s_2 to $\mathbf{N}_{\mathcal{O}}$. Therefore the target of m is $\mathcal{T}$, not $\mathcal{R}$.

Let us go to the spaces in the upper row. They are sub-ind-schemes of spaces in the lower row. So we describe the conditions defining the upper spaces in the lower spaces. The space $\mathcal{R} \times \mathcal{R}$ is clear. We impose $\varphi_{1,\mathbf{N}}(s_1) \in \mathbf{N}_{\mathcal{O}}$ in $\mathcal{T} \times \mathcal{R}$. The second space $p^{-1}(\mathcal{R} \times \mathcal{R})$ is given by the condition $\varphi_{1,\mathbf{N}} \circ \kappa_{\mathbf{N}}^{-1} \circ \varphi_{2,\mathbf{N}}(s_2) \in \mathbf{N}_{\mathcal{O}}$. For the third space $q(p^{-1}(\mathcal{R} \times \mathcal{R}))$, we need to rewrite this condition as $\varphi_{1,\mathbf{N}} \circ \eta_{\mathbf{N}}^{-1}(s_2) \in \mathbf{N}_{\mathcal{O}}$. Since q is given by setting $\eta = \varphi_2^{-1} \circ \kappa|_{D^*}$, two conditions are equivalent as required. Finally m sends $q(p^{-1}(\mathcal{R} \times \mathcal{R}))$ to $\mathcal{R}$, as $\varphi_1 \circ \eta^{-1}$ is the new trivialization of $\mathcal{P}_2$ given by the map m.

3(ii). Abstract nonsense

We need some preparatory material before we give a definition of the convolution product.

(a). Let A be a complex of sheaves on a space X. Then we have a canonical homomorphism $A \otimes \mathbb{D}A \to \boldsymbol{\omega}_X$. (See [CG97, (8.3.17)].)

(b). We define the pull-back homomorphism with support (cf. [CG97, 8.3.21]).
   Let

$$
\begin{array}{ccc}
M & \xrightarrow{\ f\ } & N \\[2pt]
\Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle j} \\[2pt]
X & \xrightarrow{\ \tilde{f}\ } & Y
\end{array}
$$

be a Cartesian square. Let A, B be complexes on N, M respectively. For a homomorphism $\varphi \in \operatorname{Hom}(A, f_* B) = \operatorname{Hom}(f^* A, B)$, we define the *pull-back with support homomorphism* in $\operatorname{Hom}(j^! A, \tilde{f}_* i^! B)$ as the composite of

$$
j^! A \to j^! f_* f^* A \cong \tilde{f}_* i^! f^* A \xrightarrow{\ \tilde{f}_* i^! \varphi\ } \tilde{f}_* i^! B,
$$

where the first map is the adjunction and the middle isomorphism is base change. It induces a homomorphism $H^*(j^! A) \to H^*(i^! B)$ on hypercohomology. We denote both homomorphisms by f^*. Note that it depends on the map f between M and N, though complexes $j^! A$, $i^! B$ are on Y, X, and the map between X and Y is $\tilde{f}$.

(c). We define an 'intersection pairing'. Let A be a complex on a space X. By 3(ii)(a) above, we have

$$
H^*(A) \otimes H^*(\mathbb{D}A) \to H^*(\boldsymbol{\omega}_X).
$$

It is constructed as follows. Let $\Delta \colon X \to X \otimes X$ be the diagonal embedding. We have $\Delta^*(A \boxtimes \mathbb{D}A) = A \otimes \mathbb{D}A$. We have the adjunction homomorphism $H^*(A) \otimes H^*(\mathbb{D}A) = H^*(A \boxtimes \mathbb{D}A) \to H^*(\Delta_* \Delta^*(A \boxtimes \mathbb{D}A)) = H^*(A \otimes \mathbb{D}A)$. We now compose 3(ii)(a).

3(iii). Convolution product

We return back to (3.2). The leftmost square is Cartesian. Thanks to the above definition, a homomorphism $p^* \boldsymbol{\omega}_{\mathcal{T} \times \mathcal{R}} \to \boldsymbol{\omega}_{G_{\mathcal{K}} \times \mathcal{R}}$ induces pull-back with support homomorphism

$$
\boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}} = (i \times \operatorname{id}_{\mathcal{R}})^! \boldsymbol{\omega}_{\mathcal{T} \times \mathcal{R}} \to \tilde{p}_* i'^! \boldsymbol{\omega}_{G_{\mathcal{K}} \times \mathcal{R}} = \tilde{p}_* \boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})}.
$$

Therefore we want to understand $p^* \boldsymbol{\omega}_{\mathcal{T} \times \mathcal{R}}$. Let us write $p = (p_{\mathcal{T}}, p_{\mathcal{R}})$ according to factors of $\mathcal{T} \times \mathcal{R}$. Then $p^* \boldsymbol{\omega}_{\mathcal{T} \times \mathcal{R}} = p_{\mathcal{T}}^* \boldsymbol{\omega}_{\mathcal{T}} \otimes p_{\mathcal{R}}^* \boldsymbol{\omega}_{\mathcal{R}}$.

Lemma 3.5. *We have isomorphisms of $G_{\mathcal{O}} \times G_{\mathcal{O}}$-equivariant complexes*

(3.6)
$$p_{\mathcal{R}}^* \boldsymbol{\omega}_{\mathcal{R}} \cong \mathbb{C}_{G_{\mathcal{K}}} \boxtimes \boldsymbol{\omega}_{\mathcal{R}},$$
$$p_{\mathcal{T}}^* \boldsymbol{\omega}_{\mathcal{T}} \cong \boldsymbol{\omega}_{G_{\mathcal{K}}} \boxtimes \mathbb{C}_{\mathcal{R}}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}],$$
$$p^* \boldsymbol{\omega}_{\mathcal{T} \times \mathcal{R}} \cong \boldsymbol{\omega}_{G_{\mathcal{K}} \times \mathcal{R}}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}].$$

Here the degree shift $[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]$ is understood by taking finite dimensional approximation as in §2(ii). The same applies to degree shifts appearing in the proof below.

Proof. The first isomorphism is obvious as $p_{\mathcal{R}}$ is just the projection to the second factor.

Note that $p_{\mathcal{T}}$ factorizes as

$$G_{\mathcal{K}} \times \mathcal{R} \xrightarrow{\operatorname{id}_{G_{\mathcal{K}}} \times \Pi} G_{\mathcal{K}} \times \mathbf{N}_{\mathcal{O}} \xrightarrow{p'_{\mathcal{T}}} \mathcal{T},$$

where $\Pi \colon \mathcal{R} \to \mathbf{N}_{\mathcal{O}}$ is the natural projection, and the second map $p'_{\mathcal{T}}$ is the quotient by $G_{\mathcal{O}}$. Since $p'_{\mathcal{T}}$ is a fiber bundle with smooth fibers, $p_{\mathcal{T}}'^* \boldsymbol{\omega}_{\mathcal{T}} \cong \boldsymbol{\omega}_{G_{\mathcal{K}}} \boxtimes \boldsymbol{\omega}_{\mathbf{N}_{\mathcal{O}}}[-2 \dim G_{\mathcal{O}}]$. Since $\mathbf{N}_{\mathcal{O}}$ is smooth, we have

$$\boldsymbol{\omega}_{\mathbf{N}_{\mathcal{O}}} = \mathbb{C}_{\mathbf{N}_{\mathcal{O}}}[2 \dim \mathbf{N}_{\mathcal{O}}].$$

We pull back further by $\operatorname{id}_{G_{\mathcal{K}}} \times \Pi$. Since the pull-back of the constant sheaf is again constant sheaf,

$$p_{\mathcal{T}}^* \boldsymbol{\omega}_{\mathcal{T}} \cong \boldsymbol{\omega}_{G_{\mathcal{K}}} \boxtimes \mathbb{C}_{\mathcal{R}}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}].$$

Finally a tensor product of any complex A with the constant sheaf is A itself. Hence we obtain (3.6). It is an isomorphism of $G_{\mathcal{O}} \times G_{\mathcal{O}}$-equivariant sheaves. □

Using (3.6) as φ in 3(ii)(b), we get the restriction with support homomorphism for sheaves and their hypercohomology groups:

(3.7)
$$p^* \colon \boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}} \to \tilde{p}_* \boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}],$$
$$p^* \colon H_*^{G_{\mathcal{O}}}(\mathcal{R}) \otimes H_*^{G_{\mathcal{O}}}(\mathcal{R}) \to H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(p^{-1}(\mathcal{R} \times \mathcal{R})).$$

Note that $G_{\mathcal{K}} \times \mathcal{R}$ is not smooth, hence it is different from the usual restriction with support (e.g., [CG97, 8.3.21]). Our definition uses a special form of p.

Let us check the degree for the second equation in (3.7). Recall the degree of $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is given relative to $2 \dim \mathbf{N}_{\mathcal{O}}$. Similarly the degree of

$H_*^{G_\mathcal{O} \times G_\mathcal{O}}(p^{-1}(\mathcal{R} \times \mathcal{R}))$ is given relative to $2 \dim \mathbf{N}_\mathcal{O} + 2 \dim G_\mathcal{O}$, as $p^{-1}(\mathcal{R} \times \mathcal{R})$ is a closed subvariety in $G_\mathcal{K} \times \mathcal{R}$, whose homology group is shifted by that. The difference of the degrees is $2 \dim \mathbf{N}_\mathcal{O} - 2 \dim G_\mathcal{O}$ appeared above. Therefore p^* preserves the degree.

Remark 3.8. Lemma 3.5 gives homomorphisms

$$p_\mathcal{T}^*\colon H_*^{G_\mathcal{O}}(\mathcal{R}) \to H_{G_\mathcal{O} \times G_\mathcal{O}}^*(i'^! A),$$
$$p_\mathcal{R}^*\colon H_*^{G_\mathcal{O}}(\mathcal{R}) \to H_{G_\mathcal{O} \times G_\mathcal{O}}^*(\mathbb{D}A),$$

where A denotes $\boldsymbol{\omega}_{G_\mathcal{K}} \boxtimes \mathbb{C}_\mathcal{R}$ for short. Then $\mathbb{C}_{G_\mathcal{K}} \boxtimes \boldsymbol{\omega}_\mathcal{R} = \mathbb{D}A$.

We compose $i'^*\colon H_{G_\mathcal{O} \times G_\mathcal{O}}^*(\mathbb{D}A) \to H_{G_\mathcal{O} \times G_\mathcal{O}}^*(i'^*\mathbb{D}A)$ with the intersection pairing in 3(ii)(c), we get

$$H_{G_\mathcal{O} \times G_\mathcal{O}}^*(i'^! A) \otimes H_{G_\mathcal{O} \times G_\mathcal{O}}^*(\mathbb{D}A) \to H_{G_\mathcal{O} \times G_\mathcal{O}}^*(\boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})}).$$

The intersection pairing is defined as $i'^*\mathbb{D}A = \mathbb{D}i'^! A$. Thus p^* is the composition of $p_\mathcal{T}^* \otimes i'^* p_\mathcal{R}^*$ and the intersection pairing.

We further have

$$H_*^{G_\mathcal{O} \times G_\mathcal{O}}(p^{-1}(\mathcal{R} \times \mathcal{R})) \xleftarrow[\cong]{\tilde{q}^*} H_*^{G_\mathcal{O}}(q(p^{-1}(\mathcal{R} \times \mathcal{R})))$$

as in (2.3). Since $\tilde{m}$ is ind-proper, we have the push-forward homomorphism for the Borel-Moore homology. We define a convolution product

$$c_1 * c_2 = \tilde{m}_*(\tilde{q}^*)^{-1} p^*(c_1 \otimes c_2) \quad \text{for } c_1, c_2 \in H_*^{G_\mathcal{O}}(\mathcal{R}).$$

The degree is preserved, so the product preserves the grading.

Remarks 3.9. (1) We have an isomorphism

$$G_\mathcal{K} \times_{G_\mathcal{O}} \mathcal{R} \xrightarrow{\cong} \mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T}; \qquad [g_1, [g_2, s]] \mapsto ([g_1, g_2 s], [g_1 g_2, s]).$$

Since $g_1(g_2 s) = (g_1 g_2)s$, it is a fiber product over $\mathbf{N}_\mathcal{K}$ as required. The map m is interpreted as the second projection p_2 from $\mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T}$, while the first projection p_1 is the descent of $p_\mathcal{T}$. (The remaining $p_\mathcal{R}$ does not seem to have a simple interpretation in terms of $\mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T}$.)

According to Remark 2.1, $\mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T}$ is analog of St. It is tempting to define a convolution product on $H_*^{G_\mathcal{K}}(\mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T})$ as usual, namely using three projections from $\mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T}$ to $\mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T}$. But we are not sure whether

even this homology group is well-defined or has a well-defined convolution product unless we identify it with $H_*^{G_O}(\mathcal{R})$. For example, we do not know how to define a convolution on *nonequivariant* homology group $H_*(\mathcal{T} \times_{\mathbf{N}_\mathcal{K}} \mathcal{T})$.

(2) There is alternative approach to a definition of the convolution product in [VV10]. The Kashiwara affine flag manifold played an essential role there. We do not check whether this approach can be applicable to our setting, nor gives the same convolution product.

(3) Our definition of the convolution product can be modified for the equivariant K-theory as follows.

First of all, the equivariant K-group of $\mathcal{R}$ is defined as the limit of $K^{G_i}(\mathcal{R}_{\leq\lambda}^d)$ as for the equivariant Borel-Moore homology group. We use the pullback with respect to $\tilde{p}_e^d \colon \mathcal{R}_{\leq\lambda}^d \to \mathcal{R}_{\leq\lambda}^e$ and pushforward with respect to the embedding $\mathcal{R}_{\leq\mu}^d \to \mathcal{R}_{\leq\lambda}^d$. These are well-defined on equivariant K-theory, as $\tilde{p}_e^d$ is flat and the embedding is proper. We now omit these d, $\leq \lambda$ from the notation and treat $\mathcal{R}$ (and other spaces $\mathcal{T}$, $G_\mathcal{O}$) as if it would be a finite dimensional scheme.

Looking back the definition of the product for homology groups, we only need to replace Lemma 3.5 and the pull-back homomorphism with support by appropriate arguments which make sense for K-theory.

Take $E \in K^{G_O}(\mathcal{R})$. We consider it as a class of an object in $D_{G_O}^b(\operatorname{Coh}(\mathcal{T}))$ whose cohomology groups are supported in $\mathcal{R}$. We replace $p_\mathcal{T}'^*(E)$ by its resolution, consisting of sheaves which are flat over $\mathbf{N}_\mathcal{O}$. This is possible since $\mathbf{N}_\mathcal{O}$ is smooth and *finite dimensional* as it is actually a truncation of the infinite dimensional $\mathbf{N}_\mathcal{O}$. Now we further pull back $p_\mathcal{T}'^*(E)$ by $\operatorname{id} \times \Pi$. It is a complex defined over $G_\mathcal{K} \times \mathcal{R}$ consisting of sheaves which are flat over $\mathcal{R}$.

Taking another $F \in D_{G_O}^b(\operatorname{Coh}(\mathcal{R}))$, we consider $p_\mathcal{R}^*(F)$, which is flat over $G_\mathcal{K}$. Then thanks to the flatness, $(\operatorname{id} \times \Pi)^*(p_\mathcal{T}'^*(E)) \otimes^L p_\mathcal{R}^*(F)$ has only finitely many higher Tor, hence a well-defined object in $D_{G_O \times G_O \times G_O}^b(\operatorname{Coh}(G_\mathcal{K} \times \mathcal{R}))$. Moreover its cohomology groups are supported on $p^{-1}(\mathcal{R} \times \mathcal{R})$.

This operation sends a distinguished triangle (either for E or F) to a distinguished triangle. Therefore descends to the equivariant K-theory.

When $\mathbf{N} = \mathfrak{g}$, the adjoint representation of G, one can easily check that this definition coincides with one in [BFM05, §7].

(4) Note that $\mathbf{N}_\mathcal{K}$ is a representation of $G_\mathcal{K}$, and $\mathbf{N}_\mathcal{O}$ is a $G_\mathcal{O}$-subrepresentation. Then $H_*^{G_O}(\mathcal{R})$ is an infinite dimensional example of *Springer theory* formulated in [Sau13], as a uniform way to look at various examples of geometric constructions of noncommutative algebras via convolution. Namely for a reductive group $\mathbf{G}$, a parabolic subgroup $\mathbf{P}$, a representation

V of $\mathbf{G}$, and a $\mathbf{P}$-subrepresentation V' of V, we introduce a variety $\mathbf{R} \overset{\text{def.}}{=}$ $\{[g, s] \in \mathbf{G} \times_P V' \mid gs \in V'\}$ with $\mathbf{P}$-action given by $p[g, s] = [pg, s]$. Then $H_*^{\mathbf{P}}(\mathbf{R})$ is an associative algebra. We take $(\mathbf{G}, \mathbf{P}, V, V') = (G_{\mathcal{K}}, G_{\mathcal{O}}, \mathbf{N}_{\mathcal{K}}, \mathbf{N}_{\mathcal{O}})$ for our $\mathcal{R}$. Another infinite dimensional examples of Springer theory, which are cousins of our $\mathcal{R}$ for quiver gauge theories, are given recently in [BEF16, Web16, §4].

Theorem 3.10. *The convolution product $*$ defines an associative graded algebra structure on $H_*^{G_{\mathcal{O}}}(\mathcal{R})$. The unit is given by the fundamental class of the fiber of $\mathcal{R} \to \mathrm{Gr}_G$ at the base point $[1] \in \mathrm{Gr}_G$. The multiplication is $H_{G_{\mathcal{O}}}^*(\mathrm{pt})$-linear in the first variable. The same assertions are true for $H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R})$.*

Since we will prove that $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is commutative, the multiplication turns out to be linear in both first and second variables. However it is not true for $H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R})$. See the computation in §3(vii)3(vii) below.

Proof of Theorem 3.10. We prove the assertions for $G_{\mathcal{O}}$ for notational simplicity. All the arguments work also for $G_{\mathcal{O}} \rtimes \mathbb{C}^\times$.

The last assertion is clear from the definition.

Let e denote the fundamental class of the fiber of $\mathcal{R} \to \mathrm{Gr}_G$ at $[1] \in \mathrm{Gr}_G$. We prove

$$e * \bullet = \mathrm{id} = \bullet * e \quad \text{on } H_*^{G_{\mathcal{O}}}(\mathcal{R}).$$

Consider $e * \bullet$. The class $(\tilde{q}^*)^{-1} p^*(e \otimes \bullet)$ is given by the pushforward homomorphism $H_*^{G_{\mathcal{O}}}(\mathcal{R}) \to H_*^{G_{\mathcal{O}}}(q(p^{-1}(\mathcal{R} \times \mathcal{R})))$ with respect to the embedding $\mathcal{R} \ni [g, s] \mapsto [\mathrm{id}, [g, s]] \in q(p^{-1}(\mathcal{R} \times \mathcal{R}))$. If we compose $\tilde{m}$, the embedding becomes just $\mathrm{id}_{\mathcal{R}}$. Therefore $e * \bullet = \mathrm{id}$. Similarly $(\tilde{q}^*)^{-1} p^*(\bullet \otimes e)$ is given by the pushforward homomorphism of the embedding $\mathcal{R} \ni [g, s] \mapsto [g, [\mathrm{id}, s]] \in q(p^{-1}(\mathcal{R} \times \mathcal{R}))$. If we compose $\tilde{m}$, it becomes $\mathrm{id}_{\mathcal{R}}$ again. Hence $\bullet * e = \mathrm{id}$.

It remains to prove the associativity.

We consider the following commutative diagram, which is a 'product' of two copies of the upper row of (3.2):

(3.11)

$$\begin{array}{ccccccc}
\mathcal{R} \times \mathcal{R} & \xleftarrow{\tilde{p}} & p^{-1}(\mathcal{R} \times \mathcal{R}) & \xrightarrow{\tilde{q}} & q(p^{-1}(\mathcal{R} \times \mathcal{R})) & \xrightarrow{\tilde{m}} & \mathcal{R} \\
\uparrow{\scriptstyle \tilde{m} \times \mathrm{id}_\mathcal{R}} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \tilde{m}} \\
q(p^{-1}(\mathcal{R} \times \mathcal{R})) \times \mathcal{R} & \longleftarrow & \boxed{3} & \longrightarrow & \boxed{4} & \longrightarrow & q(p^{-1}(\mathcal{R} \times \mathrm{id}_\mathcal{R})) \\
\uparrow{\scriptstyle \tilde{q} \times \mathrm{id}_\mathcal{R}} & & \uparrow & & \uparrow & & \uparrow{\scriptstyle \tilde{q}} \\
p^{-1}(\mathcal{R} \times \mathcal{R}) \times \mathcal{R} & \longleftarrow & \boxed{1} & \longrightarrow & \boxed{2} & \longrightarrow & p^{-1}(\mathcal{R} \times \mathcal{R}) \\
\downarrow{\scriptstyle \tilde{p} \times \mathrm{id}_\mathcal{R}} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \tilde{p}} \\
\mathcal{R} \times \mathcal{R} \times \mathcal{R} & \xleftarrow{\mathrm{id}_\mathcal{R} \times \tilde{p}} & \mathcal{R} \times p^{-1}(\mathcal{R} \times \mathcal{R}) & \xrightarrow{\mathrm{id}_\mathcal{R} \times \tilde{q}} & \mathcal{R} \times q(p^{-1}(\mathcal{R} \times \mathcal{R})) & \xrightarrow{\mathrm{id}_\mathcal{R} \times \tilde{m}} & \mathcal{R} \times \mathcal{R},
\end{array}$$

where

$$\boxed{1} = \{(g_1, g_2, [g_3, s]) \in G_\mathcal{K} \times G_\mathcal{K} \times \mathcal{R} \mid g_2 g_3 s, g_1 g_2 g_3 s \in \mathbf{N}_\mathcal{O}\},$$

and $\boxed{2}$, $\boxed{3}$, $\boxed{4}$ are quotients of $\boxed{1}$ by $1 \times G_\mathcal{O}$, $G_\mathcal{O} \times 1$, $G_\mathcal{O} \times G_\mathcal{O}$ respectively. Here $G_\mathcal{O} \times G_\mathcal{O}$ acts on $\boxed{1}$ by

$$(h_1, h_2) \cdot (g_1, g_2, [g_3, s]) = (g_1 h_1^{-1}, h_1 g_2 h_2^{-1}, [h_2 g_3, s])$$
$$\text{for } (h_1, h_2) \in G_\mathcal{O} \times G_\mathcal{O}.$$

Horizontal and vertical arrows from $\boxed{1}$, $\boxed{4}$ are given by
(3.12)

$$(g_1, [g_2, g_3 s], [g_3, s]) \longleftarrow\!\!\!\shortmid (g_1, g_2, [g_3, s]) \in \boxed{1} \qquad\qquad [g_1 g_2, [g_3, s]]$$
$$\Big\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \Big\uparrow$$
$$([g_1, g_2 g_3 s], (g_2, [g_3, s])), \quad \boxed{4} \ni [g_1, [g_2, [g_3, s]]] \mapsto [g_1, [g_2 g_3, s]].$$

Arrows from $\boxed{2}$, $\boxed{3}$ are given by trivial modification of above ones, as $\boxed{1} \to \boxed{3}$, etc. are fiber bundles.

The convolution product $c_1 * (c_2 * c_3)$ is given by applying induced homomorphisms in the bottom row from left to right, and then going up in the rightmost column. Similarly $(c_1 * c_2) * c_3$ is given by going the leftmost column and the top row. Therefore the associativity follows if we show that arrows induce appropriate pull-back or push-forward homomorphisms, and they form a commutative diagram for each square.

Let us first look at the bottom left square. We can extend the square to a cube as

$$
\begin{array}{c}
\xymatrix{
& G_{\mathcal{K}} \times \mathcal{R} \times \mathcal{R} & & G_{\mathcal{K}} \times p^{-1}(\mathcal{R} \times \mathcal{R}) \\
p^{-1}(\mathcal{R} \times \mathcal{R}) \times \mathcal{R} & & \boxed{1} & \\
& \mathcal{T} \times \mathcal{R} \times \mathcal{R} & & \mathcal{T} \times p^{-1}(\mathcal{R} \times \mathcal{R}) \\
\mathcal{R} \times \mathcal{R} \times \mathcal{R} & & \mathcal{R} \times p^{-1}(\mathcal{R} \times \mathcal{R}) &
}
\end{array}
$$

with arrows labeled $\mathrm{id}_{G_{\mathcal{K}}} \times \tilde{p}$, $p \times \mathrm{id}_{\mathcal{R}}$, P, $\mathrm{id}_{\mathcal{T}} \times \tilde{p}$.

Arrows from spaces in the front square to those in the rear square are closed embeddings. Arrows in the rear square are as indicated, and one remaining $P \colon G_{\mathcal{K}} \times p^{-1}(\mathcal{R} \times \mathcal{R}) \to \mathcal{T} \times p^{-1}(\mathcal{R} \times \mathcal{R})$ is given by the formula for the corresponding map in the front square in (3.12).

Homomorphisms between dualizing complexes $\boldsymbol{\omega}$ and their pull-backs have been already constructed for p and $\tilde{p}$ in (3.6) and (3.7) respectively. For P, we decompose $P = (P_{\mathcal{T}}, P_{p^{-1}(\mathcal{R} \times \mathcal{R})})$ and see that

$$
P^*_{p^{-1}(\mathcal{R} \times \mathcal{R})} \boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})} \cong \mathbb{C}_{G_{\mathcal{K}}} \boxtimes \boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})},
$$
$$
P^*_{\mathcal{T}} \boldsymbol{\omega}_{\mathcal{T}} \cong \boldsymbol{\omega}_{G_{\mathcal{K}}} \boxtimes \mathbb{C}_{p^{-1}(\mathcal{R} \times \mathcal{R})}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}],
$$

as in Lemma 3.5. We then construct pull-back homomorphisms with support (3(ii)(b)) for the front square, as the top and bottom squares are cartesian.

In order to show commutativity of pull-back homomorphisms, it is enough to consider the rear square by the construction of pull-back homomorphisms with support. Let us factor $\boldsymbol{\omega}_{\mathcal{T} \times \mathcal{R} \times \mathcal{R}} = \boldsymbol{\omega}_{\mathcal{T}} \boxtimes \boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}}$ as before, and consider the pull-backs of $\boldsymbol{\omega}_{\mathcal{T}}$ and $\boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}}$ separately.

Let us first consider $\boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}}$. We have two homomorphisms

$$
\begin{array}{ccc}
P^*(\mathrm{id}_{\mathcal{T}} \times \tilde{p})^*(\mathbb{C}_{\mathcal{T}} \boxtimes \boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}}) & \longrightarrow & \mathbb{C}_{G_{\mathcal{K}}} \boxtimes \boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}] \\
\| & & \| \\
(p \times \mathrm{id}_{\mathcal{R}})^*(\mathrm{id}_{G_{\mathcal{K}}} \times \tilde{p})^*(\mathbb{C}_{\mathcal{T}} \boxtimes \boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}}) & \longrightarrow & \mathbb{C}_{G_{\mathcal{K}}} \boxtimes \boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]
\end{array}
$$

by following left, top arrows and bottom, right arrows. They are the same, as both are essentially given by $p^* \colon \tilde{p}^* \boldsymbol{\omega}_{\mathcal{R} \times \mathcal{R}} \to \boldsymbol{\omega}_{p^{-1}(\mathcal{R} \times \mathcal{R})}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}]$ constructed in (3.7).

Next consider $\boldsymbol{\omega}_{\mathcal{T}}$. The $\mathcal{T}$-component of $(\mathrm{id}_{\mathcal{T}} \times \tilde{p}) \circ P = (\mathrm{id}_{G_{\mathcal{K}}} \times \tilde{p}) \circ (p \times \mathrm{id}_{\mathcal{R}})$ factors as

$$G_{\mathcal{K}} \times p^{-1}(\mathcal{R} \times \mathcal{R}) \xrightarrow{\mathrm{id}_{G_{\mathcal{K}}} \times \Pi'} G_{\mathcal{K}} \times \mathbf{N}_{\mathcal{O}} \xrightarrow{p'_{\mathcal{T}}} \mathcal{T},$$

where $\Pi' \colon p^{-1}(\mathcal{R} \times \mathcal{R}) \to \mathbf{N}_{\mathcal{O}}$ is $(g_2, [g_3, s]) \mapsto g_2 g_3 s$. As in the proof of Lemma 3.5, we have

$$((\mathrm{id}_{\mathcal{T}} \times \tilde{p}) \circ P)^*(\boldsymbol{\omega}_{\mathcal{T}} \boxtimes \mathbb{C}_{\mathcal{R} \times \mathcal{R}}) \cong \boldsymbol{\omega}_{G_{\mathcal{K}}} \boxtimes \mathbb{C}_{p^{-1}(\mathcal{R} \times \mathcal{R})}[2 \dim \mathbf{N}_{\mathcal{O}} - 2 \dim G_{\mathcal{O}}].$$

Two homomorphisms which are constructed by going along left, top arrows and bottom, right arrows are the same, as they are constructed in the same way. This completes the proof of the commutativity at the bottom left square.

Since $\tilde{q} \colon p^{-1}(\mathcal{R} \times \mathcal{R}) \to q(p^{-1}(\mathcal{R} \times \mathcal{R}))$ is a fiber bundle with fibers $G_{\mathcal{O}}$, commutativity for squares involving $\tilde{q}$ is obvious. Let us consider the right bottom square. We extend it to a cube:

Arrows from the front to rear are closed embeddings. The map $P' \colon G_{\mathcal{K}} \times q(p^{-1}(\mathcal{R} \times \mathcal{R})) \to \mathcal{T} \times q(p^{-1}(\mathcal{R} \times \mathcal{R}))$ is given by the formula in (3.12). Recall that the pull-back with support homomorphism p^* from $\mathcal{R} \times \mathcal{R}$ to $p^{-1}(\mathcal{R} \times \mathcal{R})$ is defined via p. Similarly the pull-back from $\mathcal{R} \times q(p^{-1}(\mathcal{R} \times \mathcal{R}))$ to $\boxed{2}$ is defined via P'. Therefore it is enough to check the commutativity in the rear square, i.e.,

$$p^*(\mathrm{id}_{\mathcal{T}} \times \tilde{m})_! = (\mathrm{id}_{G_{\mathcal{K}}} \times \tilde{m})_! P'^* \colon H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(\mathcal{T} \times q(p^{-1}(\mathcal{R} \times \mathcal{R})))$$
$$\to H_*^{G_{\mathcal{O}} \times G_{\mathcal{O}}}(G_{\mathcal{K}} \times \mathcal{R})$$

(with appropriate degree shift). But this follows from the base change, as the rear square is cartesian.

The commutativity of the right top square is clear, as it involves only pushforward homomorphisms.   □

3(iv). Definition of the Coulomb branch

Let us denote $(H_*^{G_{\mathcal{O}}}(\mathcal{R}), *)$ by $\mathcal{A}$ or $\mathcal{A}(G, \mathbf{N})$ when we want to emphasize $(G, \mathbf{N})$. We will prove that $\mathcal{A}$ is commutative later. Therefore we can consider its spectrum. Here is our main proposal:

Definition 3.13. We define the *Coulomb branch* $\mathcal{M}_C$ (as an affine scheme) by

$$\mathcal{M}_C \equiv \mathcal{M}_C(G, \mathbf{N}) \stackrel{\text{def.}}{=} \operatorname{Spec} \mathcal{A}.$$

If we add the loop rotation, the equivariant homology group $(H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^{\times}}(\mathcal{R}), *)$ is a noncommutative deformation of the Coulomb branch $\mathcal{M}_C$. Let us denote it by $\mathcal{A}_{\hbar}$ or $\mathcal{A}_{\hbar}(G, \mathbf{N})$. We call it the *quantized Coulomb branch*. In particular, $\mathcal{M}_C$ has a natural Poisson structure.

We will show that $\mathcal{A}$ is finitely generated and integral later (see Proposition 6.8, Corollary 5.22). We also prove that $\mathcal{A}$ is normal (Proposition 6.12). It could be compatible with the following example in [CHMZ15, Table 10]: Take a nilpotent orbit $\mathcal{O}$ whose Lusztig-Spaltenstein dual (in the Langlands dual Lie algebra) is not normal. Take a gauge theory whose Higgs branch is the intersection of the nilpotent cone and the Slodowy slice to $\mathcal{O}$. Such a gauge theory exists for classical groups (see [Nak16, Appendix A]). A naive guess gives us Lusztig-Spaltenstein dual of $\mathcal{O}$ as the Coulomb branch, hence is not normal. But [CHMZ15, Table 10] suggests us the normalization of the Lusztig-Spaltenstein dual instead. Note however that they are *not* of cotangent type, hence our construction does not apply. Therefore it is a little early to make any conclusion, but it seems natural to conjecture that $\mathcal{A}$ is normal even if not necessarily of cotangent type.

In many examples, $\mathcal{A}$ is Cohen-Macaulay. We will use these properties crucially in [Quiver, Theorem 3.1] and we do not have counter-examples at this moment. Moreover we show that the Poisson structure is symplectic on the regular locus of $\mathcal{M}_C$ (Proposition 6.15) and believe that there is a hyper-Kähler structure there. This is what physicists have expected. We optimistically conjecture that $\mathcal{M}_C$ has only *symplectic singularities* [Bea00], i.e., the symplectic form on the smooth locus extends to a holomorphic 2-form on a resolution $\tilde{\mathcal{M}}_C \to \mathcal{M}_C$.

We will give two proofs of the commutativity of $\mathcal{A}$. The first proof is a reduction to the abelian case, and is given in Theorem 4.1 and Proposition 5.15.

The second proof is a well-known argument: Using the Beilinson-Drinfeld Grassmannian to deform a situation where the product $c_1 * c_2$ is manifestly symmetric under $c_1 \leftrightarrow c_2$. Then we use nearby cycle functors and (dual) *specialization* homomorphism. In fact, we will prove a commutativity at the level of an object in the $G_{\mathcal{O}}$-equivariant derived category of constructible sheaves on Gr_G. See [Affine, §§2, 3] for more detail.

Remark 3.14. In the same way we define the *K-theoretic Coulomb branch* as $\mathcal{M}_C^K \overset{\mathrm{def.}}{=} \mathrm{Spec}\, K^{G_{\mathcal{O}}}(\mathcal{R})$ thanks to Remark 3.9(3). The first proof of the commutativity is a reduction to abelian cases, and hence works for $K^{G_{\mathcal{O}}}(\mathcal{R})$. (The second proof seems only applicable for homology groups.) The proofs that $\mathcal{A}$ is finitely generated and integral work for $K^{G_{\mathcal{O}}}(\mathcal{R})$. The proof of the normality does not work, as we will use the grading in Lemma 6.9(1).

3(v). Grading and group action

(See [Nak16, §4(iv), Properties (a,c)] for original sources in physics.)

Recall that connected components of Gr_G are parametrized by $\pi_1(G)$. Since $\mathcal{R}$ is homotopy equivalent to Gr_G, we also have $\pi_0(\mathcal{R}) \cong \pi_1(G)$. Thus we have a decomposition

$$H_*^{G_{\mathcal{O}}}(\mathcal{R}) \cong \bigoplus_\gamma H_*^{G_{\mathcal{O}}}(\mathcal{R}^\gamma),$$

where $\mathcal{R}^\gamma$ is the connected component corresponding to $\gamma \in \pi_1(G)$. This decomposition is compatible with the convolution product:

$$H_*^{G_{\mathcal{O}}}(\mathcal{R}^{\gamma_1}) * H_*^{G_{\mathcal{O}}}(\mathcal{R}^{\gamma_2}) \subset H_*^{G_{\mathcal{O}}}(\mathcal{R}^{\gamma_1+\gamma_2}),$$

where $\gamma_1 + \gamma_2$ is the sum of γ_1 and γ_2 in the abelian group $\pi_1(G)$. Therefore $\mathcal{A}$ is a $\mathbb{Z} \times \pi_1(G)$-graded algebra, where the first $\mathbb{Z}$ is the half of the cohomological grading. (The odd degree part vanishes by Proposition 2.7.) This gives an action of $\mathbb{C}^\times \times \pi_1(G)^\wedge$ on the spectrum $\mathcal{M}_C$, where $\pi_1(G)^\wedge$ is the Pontryagin dual of $\pi_1(G)$.

3(vi). Cartan subalgebra – a commutative subalgebra in the quantized Coulomb branch

Recall that $\mathcal{A}_\hbar = H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R})$ is a module over $H_{G \times \mathbb{C}^\times}^*(\mathrm{pt})$. Let T be a maximal torus of G with the Lie algebra $\mathfrak{t}$. Let W be the Weyl group. We have $H_G^*(\mathrm{pt}) \cong \mathbb{C}[\mathfrak{t}]^W$. We add a variable $\hbar$ for the $\mathbb{C}^\times$-part, so $H_{G \times \mathbb{C}^\times}^*(\mathrm{pt}) \cong \mathbb{C}[\hbar, \mathfrak{t}]^W$. It is also known that $\mathfrak{t}/W \cong \mathbb{C}^\ell$, where ℓ is the rank of G.

Proposition 3.15. *Let 1 be the unit of the algebra $\mathcal{A}_\hbar$. Then $H_{G \times \mathbb{C}^\times}^*(\mathrm{pt})1 = \mathbb{C}[\hbar, \mathfrak{t}]^W 1$ forms a commutative subalgebra of $\mathcal{A}_\hbar$.*

Definition 3.16. We call the commutative subalgebra $\mathbb{C}[\hbar, \mathfrak{t}]^W 1$ the *Cartan subalgebra*[2] of the quantized Coulomb branch.

Proof of Proposition 3.15. It can be checked directly, but this is a formal consequence of properties that have been already established in Theorem 3.10. Let c_1, $c_2 \in H_{G \times \mathbb{C}^\times}^*(\mathrm{pt})$. Then

$$(c_1 1) * (c_2 1) = c_1(1 * (c_2 1)) = c_1(c_2 1) = (c_1 c_2)1.$$

The first equality is the linearity of the multiplication $*$ with respect to the first variable. The second equality holds as 1 is unit. The third is true, as $H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R})$ is a module over $H_{G \times \mathbb{C}^\times}^*(\mathrm{pt})$.

Since $H_{G \times \mathbb{C}^\times}^*(\mathrm{pt})$ is commutative, the above implies the assertion. $\square$

By specializing Proposition 3.15 at $\hbar = 0$, we have a Poisson commuting subalgebra $H_G^*(\mathrm{pt}) \cong \mathbb{C}[\mathfrak{t}]^W$ of $\mathcal{A}$. Therefore we have a morphism

$$(3.17) \qquad \varpi \colon \mathcal{M}_C = \mathrm{Spec}\,\mathcal{A} \to \mathrm{Spec}(H_G^*(\mathrm{pt})) \cong \mathfrak{t}/W \cong \mathbb{C}^\ell,$$

and all the functions factoring through ϖ are Poisson commuting. It will be proved in Proposition 5.19 that the generic fiber is $T^\vee$, the dual torus of T. Hence ϖ is a complete integrable system.

The Poisson bracket $\{\ ,\ \}$ is the commutator $[\ ,\ ]$ divided by $\hbar$. Therefore it is of degree $- \deg \hbar = -1$.

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[2]This commutative subalgebra is called *Gelfand-Tsetlin algebra* in related contexts. But our subalgebra is hardly worth this name, as the proof of the commutativity is just a tautology. This alternative name is proposed to us by Boris Feigin.

3(vii). Trivial properties of Coulomb branches

Let us list a few trivial properties of $\mathcal{M}_C$ and $\mathcal{A}_\hbar$, which follow directly from the definition.

(a). Suppose $(G, \mathbf{N}) = (G_1 \times G_2, \mathbf{N}_1 \oplus \mathbf{N}_2)$, where $\mathbf{N}_i$ is a representation of G_i ($i = 1, 2$). Then

$$\mathcal{M}_C(G, \mathbf{N}) = \mathcal{M}_C(G_1, \mathbf{N}_1) \times \mathcal{M}_C(G_2, \mathbf{N}_2),$$
$$\mathcal{A}_\hbar(G, \mathbf{N}) = \mathcal{A}_\hbar(G_1, \mathbf{N}_1) \otimes_{\mathbb{C}[\hbar]} \mathcal{A}_\hbar(G_2, \mathbf{N}_2).$$

These follow from $\mathcal{R}_{G,\mathbf{N}} = \mathcal{R}_{G_1,\mathbf{N}_1} \times \mathcal{R}_{G_2,\mathbf{N}_2}$, and the Künneth formula $H_*^{G_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}}) = H_*^{(G_1)_\mathcal{O}}(\mathcal{R}_{G_1,\mathbf{N}_1}) \otimes_\mathbb{C} H_*^{(G_2)_\mathcal{O}}(\mathcal{R}_{G_2,\mathbf{N}_2})$ and its $\mathbb{C}^\times$-equivariant version.

(b). Suppose $(G, \mathbf{N}) = (G, \mathbf{N}_1 \oplus \mathbf{N}')$, where $\mathbf{N}'$ is a trivial representation of G. Then

$$\mathcal{M}_C(G, \mathbf{N}) = \mathcal{M}_C(G, \mathbf{N}_1), \qquad \mathcal{A}_\hbar(G, \mathbf{N}) = \mathcal{A}_\hbar(G, \mathbf{N}_1).$$

Take $G_1 = G$, $G_2 = \{e\}$, $\mathbf{N}_2 = \mathbf{N}'$ in 3(vii). We have $\mathrm{Gr}_{\{1\}}$ is just a single point and $\mathcal{R}_{\{e\},\mathbf{N}'} \cong \mathbf{N}'_\mathcal{O}$. Its homology $H_*(\mathbf{N}'_\mathcal{O})$ is spanned by the fundamental class of $\mathbf{N}'_\mathcal{O}$. We have $H_*(\mathbf{N}'_\mathcal{O}) \cong \mathbb{C}$, as an algebra. Therefore $\mathcal{M}_C(\{1\}, \mathbf{N}')$ is a single point. For the quantized version, we have $H_*^{\mathbb{C}^\times}(\mathbf{N}'_\mathcal{O}) \cong \mathbb{C}[\hbar]$.

(c). Let $G' \to G$ be a finite covering, and let $\pi_1(G') \subset \pi_1(G)$ be the corresponding cofinite subgroup of $\pi_1(G)$. Let Γ be the Pontryagin dual of $\pi_1(G)/\pi_1(G')$, considered as a subgroup of $\pi_1(G)^\wedge$. It acts on $\mathcal{M}_C(G, \mathbf{N})$ by the construction in §3(v). Let us consider $\mathbf{N}$ as a representation of G' through the projection $G' \to G$. Then

$$\mathcal{M}_C(G', \mathbf{N}) = \mathcal{M}_C(G, \mathbf{N})/\Gamma, \qquad \mathcal{A}_\hbar(G', \mathbf{N}) = \mathcal{A}_\hbar(G, \mathbf{N})^\Gamma.$$

It is known that $\mathrm{Gr}_{G'}$ is the union of components of Gr_G corresponding to $\pi_1(G') \subset \pi_1(G) \cong \pi_0(\mathrm{Gr}_G)$. (See e.g., [BeiDr, 4.5.6].) The same is true for $\mathcal{R}_{G',\mathbf{N}}$. Note also that there is no difference between equivariant homology groups for $G_\mathcal{O}$ and $G'_\mathcal{O}$ as we consider over complex coefficients. Therefore $H_*^{G'_\mathcal{O}}(\mathcal{R}_{G',\mathbf{N}})$ is just the Γ-invariant part of $H_*^{G_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}})$. It means the assertion.

(d). Next one is similar to the above, but the case when the Pontryagin dual of $\pi_1(\tilde{G})/\pi_1(G)$ is a torus. Let $1 \to G \to \tilde{G} \to T_F \to 1$ be a short exact sequence of connected reductive groups where T_F is a torus. The subscript 'F' stands for *flavor* symmetry that will be discussed more generally in §§3(viii), 3(ix). For any representation $\mathbf{N}$ of $\tilde{G}$ we can consider the corresponding Coulomb branch $\mathcal{M}_C(\tilde{G}, \mathbf{N})$. It acquires an action of the dual torus $T_F^\vee = \pi_1(T_F)^\wedge$ by §3(v). Then

Proposition 3.18.

$$\mathcal{M}_C(G, \mathbf{N}) \cong \textit{Hamiltonian reduction of } \mathcal{M}_C(\tilde{G}, \mathbf{N}) \textit{ by } T_F^\vee.$$

Since $\mathcal{M}_C(\tilde{G}, \mathbf{N})$ has singularities in general, this statement means an algebraic counterpart, i.e., $\mathcal{A}(G, \mathbf{N})$ is the $T_F^\vee$-invariant part of $\mathcal{A}(\tilde{G}, \mathbf{N})/\{\mu_{T_F^\vee} = 0\}$, where $\mu_{T_F^\vee}$ is the moment map for the $T_F^\vee$-action, which is described as follows.

Let $\mathfrak{t}_F = \operatorname{Lie} T_F \cong (\operatorname{Lie} T_F^\vee)^*$. Recall that a map $\mu_{T_F^\vee} \colon \mathcal{M}_C(\tilde{G}, \mathbf{N}) \to \mathfrak{t}_F$ is a moment map if $\xi \circ \mu_{T_F^\vee}$ is a hamiltonian for a vector field ξ^* generated by $\xi \in \operatorname{Lie} T_F^\vee = \mathfrak{t}_F^*$. It is equivalent to say that the Poisson bracket satisfies $\{f, \xi \circ \mu_{T_F^\vee}\} = \xi^*(f)$ for any function $f \in \mathcal{A}(\tilde{G}, \mathbf{N})$. This notion makes sense for Poisson algebras. A moment map is not unique in general, but we have the canonical one given by the composite of $\mathcal{M}_C(\tilde{G}, \mathbf{N}) \xrightarrow{\varpi} \operatorname{Spec} H_{\tilde{G}}^*(\mathrm{pt}) \to \operatorname{Spec} H_{T_F}^*(\mathrm{pt})$, or in other words:

Lemma 3.19. *The composite of*

$$\mathbb{C}[\mathfrak{t}_F] = H_{T_F}^*(\mathrm{pt}) \to H_{\tilde{G}}^*(\mathrm{pt}) \xrightarrow{\varpi^*} H_*^{\tilde{G}\circ}(\mathcal{R}_{\tilde{G},\mathbf{N}}) = \mathcal{A}(\tilde{G}, \mathbf{N})$$

is the comoment map, i.e., the pull-back by the moment map $\mu_{T_F^\vee} \colon \mathcal{M}_C(\tilde{G}, \mathbf{N}) \to \mathfrak{t}_F$. Here the first homomorphism is induced by $\tilde{G} \to T_F$, and the second one is multiplication to the unit $1 \in H_*^{\tilde{G}\circ}(\mathcal{R}_{\tilde{G},\mathbf{N}})$ as in Proposition 3.15, which is the pull-back by ϖ in (3.17).

Proof. We take a character $\chi \colon T_F \to \mathbb{C}^\times$ and consider its first Chern class $c_1(\chi)$ as an element in $H_{T_F}^2(\mathrm{pt})$. We also consider it as $\tilde{G} \to \mathbb{C}^\times$, and hence as an element in $H_{\tilde{G}}^2(\mathrm{pt})$. We have the induced $\mathbb{C}^\times$-action through $\mathbb{C}^\times = (\mathbb{C}^\times)^\vee \to T_F^\vee$. We need to show that the comoment map for $\mathbb{C}^\times$ is $c_1(\chi) \mapsto c_1(\chi)1 \in H_*^{\tilde{G}\circ}(\mathcal{R}_{\tilde{G},\mathbf{N}})$.

We consider the quantized version of the homomorphism

$$H^*_{\tilde{G} \times \mathbb{C}^\times}(\mathrm{pt}) \to H^{\tilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times}_*(\mathcal{R}_{\tilde{G}, \mathbf{N}})$$

and the commutator $[\bullet, c_1(\chi)1]$. The assertion follows from the following lemma below.

Indeed, it implies that the Poisson bracket $\{\bullet, c_1(\chi)1\}$ is $\pi_1(\chi)\,\mathrm{id}$, which is the action of the Lie algebra. This is nothing but the definition of the comoment map. □

Lemma 3.20. *Let $\chi\colon \tilde{G} \to \mathbb{C}^\times$ be a character and consider its first Chern class $c_1(\chi)$ as an element in $H^2_{\tilde{G}}(\mathrm{pt})$. Then $[\bullet, c_1(\chi)1] = \hbar\pi_1(\chi)\,\mathrm{id}$, where $\pi_1(\chi)$ is the $\mathbb{Z}$-valued function given by $\pi_0(\mathcal{R}_{\tilde{G}, \mathbf{N}}) = \pi_1(\tilde{G}) \xrightarrow{\pi_1(\chi)} \pi_1(\mathbb{C}^\times) = \mathbb{Z}$.*

Proof. By the definition of the convolution product, $[\bullet, c_1(\chi)1]$ is given by the cup product with respect to the first Chern class of the line bundle over $\mathcal{R}_{\tilde{G}, \mathbf{N}}$ induced from the composite of $\tilde{G}_\mathcal{O} \to \tilde{G} \xrightarrow{\chi} \mathbb{C}^\times$, where the first homomorphism is given by taking the constant term. The line bundle is the pull-back from $\mathrm{Gr}_{\tilde{G}}$ by the projection $\mathcal{R}_{\tilde{G}, \mathbf{N}} \to \mathrm{Gr}_{\tilde{G}}$. It is further the pull-back from $\mathrm{Gr}_{\mathbb{C}^\times}$ by the morphism $\mathrm{Gr}_{\tilde{G}} \to \mathrm{Gr}_{\mathbb{C}^\times}$ given by χ. Let us note that the identification $\mathrm{Gr}_{\mathbb{C}^\times} \simeq \mathbb{Z}$ is given by $\mathbb{C}^\times_\mathcal{K}/\mathbb{C}^\times_\mathcal{O} \ni [z^n] \mapsto n$. Then the line bundle is trivial, equipped with the $\mathbb{C}^\times_\mathcal{O} \rtimes \mathbb{C}^\times$-equivariant structure by the nth power map $\mathbb{C}^\times \to \mathbb{C}^\times$ on the component for $n \in \mathbb{Z}$.

Now $\pi_1(\chi)$ is given by $\mathcal{R}_{\tilde{G}, \mathbf{N}} \to \mathrm{Gr}_{\mathbb{C}^\times}$, and $[\bullet, c_1(\chi)1] = \hbar\pi_1(\chi)\,\mathrm{id}$ follows. □

Proof of Proposition 3.18. Since T_F is torus, its affine Grassmannian is a discrete lattice. Hence $\mathcal{R}_{G, \mathbf{N}}$ is a union of components of $\mathcal{R}_{\tilde{G}, \mathbf{N}}$ as in 3(vii). Hence $H^{\tilde{G}_\mathcal{O}}_*(\mathcal{R}_{G, \mathbf{N}})$ is the $T^\vee_F$-invariant part of $H^{\tilde{G}_\mathcal{O}}_*(\mathcal{R}_{\tilde{G}, \mathbf{N}})$. Next the equivariant homology for $G_\mathcal{O}$ is given by

$$H^{\tilde{G}_\mathcal{O}}_*(\mathcal{R}_{G, \mathbf{N}}) \otimes_{H^*_{\tilde{G}}(\mathrm{pt})} H^*_G(\mathrm{pt}) = H^{\tilde{G}_\mathcal{O}}_*(\mathcal{R}_{G, \mathbf{N}}) \otimes_{H^*_{T_F}(\mathrm{pt})} \mathbb{C}.$$

This is given by cutting out the ideal generated by $c_1(\chi)1$ for various $\chi \in H^2_{T_F}(\mathrm{pt})$.

We need to check that the induced product and the original product on $H^{G_\mathcal{O}}_*(\mathcal{R}_{G, \mathbf{N}})$ are equal. Namely both maps

$$H^{\tilde{G}_\mathcal{O}}_*(\mathcal{R}_{\tilde{G}, \mathbf{N}}) \longleftarrow H^{\tilde{G}_\mathcal{O}}_*(\mathcal{R}_{G, \mathbf{N}}) \longrightarrow H^{G_\mathcal{O}}_*(\mathcal{R}_{G, \mathbf{N}})$$

are algebra homomorphisms. The left arrow is the embedding of the degree 0 part with respect to the grading given by $\pi_1(T_F)$ in §3(v), and hence it is an algebra embedding. The right arrow will be studied in more general setting in Proposition 3.22. It will be shown that it is an algebra homomorphism. Moreover the multiplication on $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}})$ in Proposition 3.22 is the one given as a subalgebra: The diagram (3.23) is the degree 0 part of the diagram (3.2) for $\mathcal{R}_{\tilde{G},\mathbf{N}}$. Therefore $\mathcal{M}_C(G, \mathbf{N})$ is the Hamiltonian reduction of $\mathcal{M}_C(\tilde{G}, \mathbf{N})$ by $T_F^\vee$. □

Now the corresponding statement for quantized Coulomb branches is clear.

$\mathcal{A}_\hbar(G, \mathbf{N})$ is the quantum Hamiltonian reduction of $\mathcal{A}_\hbar(\tilde{G}, \mathbf{N})$ by $T_F^\vee$.

Recall a homomorphism $\mu^* \colon U(\operatorname{Lie} T_F^\vee)[\hbar] \to \mathcal{A}_\hbar(\tilde{G}, \mathbf{N})$ is a *quantum comoment map* if $[f, \mu^*(\xi)] = \hbar\xi^*(f)$ for $\xi \in \operatorname{Lie} T_F^\vee$, $f \in \mathcal{A}_\hbar(\tilde{G}, \mathbf{N})$. Since $T_F^\vee$ is torus, we have

$$U(\operatorname{Lie} T_F^\vee) \cong S(\operatorname{Lie} T_F^\vee) \cong \mathbb{C}[\mathfrak{t}_F].$$

The above proof of Lemma 3.19, in fact, shows that the composite of $\mathbb{C}[\mathfrak{t}, \hbar] = H_{T_F \times \mathbb{C}^\times}^*(\mathrm{pt}) \to H_{\tilde{G} \times \mathbb{C}^\times}^*(\mathrm{pt}) \to \mathcal{A}_\hbar(\tilde{G}, \mathbf{N})$ is a quantum comoment map. Furthermore $H_*^{\tilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}_{G,\mathbf{N}}) = \mathcal{A}_\hbar(\tilde{G}, \mathbf{N})^{T_F^\vee}$ and $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}_{G,\mathbf{N}}) = H_*^{\tilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}_{G,\mathbf{N}}) \otimes_{H_{\tilde{G} \times \mathbb{C}^\times}^*(\mathrm{pt})} H_{G \times \mathbb{C}^\times}^*(\mathrm{pt})$. Hence $\mathcal{A}_\hbar(G, \mathbf{N})$ is the quotient of $\mathcal{A}_\hbar(\tilde{G}, \mathbf{N})^{T_F^\vee}$ by the intersection of $\mathcal{A}_\hbar(\tilde{G}, \mathbf{N})^{T_F^\vee}$ and the right ideal generated by the image of the quantum comoment map. (The intersection is a two-sided ideal, and the quotient is actually a ring.) This is nothing but the definition of the quantum Hamiltonian reduction. See [Eti07, §4].

Example 3.21. Let us give an example of this construction. Let $(\tilde{G}, \mathbf{N}) = (\mathrm{GL}(2), (\mathbb{C}^2)^{\oplus N_f})$, and $G = \mathrm{SL}(2)$. Here $\mathbb{C}^2$ is the vector representation of $\mathrm{GL}(2)$ and $(\mathbb{C}^2)^{\oplus N_f}$ is the direct sum of its N_f copies. Then $(\tilde{G}, \mathbf{N})$ is a quiver gauge theory of type A_1 with $\dim V = 2$, $\dim W = N_f$. Assume $N_f \geq 4$ so that the corresponding vector is dominant. As we will prove in [Quiver, §3(iii)], the Coulomb branch for $(\tilde{G}, \mathbf{N})$ is a quiver variety of type $A_{N_f - 1}$ with dimension vectors $\dim V = (1, 2, \ldots, 2, 1)$, $\dim W = (0, 1, 0, \ldots, 0, 1, 0) \in \mathbb{Z}^{N_f - 1}$. Moreover, the torus $\pi_1(\tilde{G})^\vee$ action is identified with the action of $\mathrm{GL}(W_2) \cong \mathbb{C}^\times$. (See [Quiver, Remark 3.12].) Therefore the Coulomb branch of $(G, (\mathbb{C}^2)^{N_f})$ is the Hamiltonian reduction of $\prod \mathrm{GL}(V_i) \times \mathrm{GL}(W_2)$, which is a quiver variety of type D_{N_f} of dimension vectors $\dim V = (\overset{1}{\underset{1}{2}} \ldots 21)$, $\dim W = (\overset{0}{\underset{0}{0}} \ldots 010) \in \mathbb{Z}^{N_f}$. This quiver variety is Kronheimer's original

construction of a simple singularity of type D_{N_f}. This coincides with the expectation in [SW97].

By [SW97, (2.18)], $\mathcal{M}_C(G, \mathbf{N})$ is in general expected to be $y^2 = x^2 v - v^{N_f - 1}$ even for $N_f = 1, 2, 3$. In this case, $\mathcal{M}_C(\tilde{G}, \mathbf{N})$ is *not* a quiver variety, hence the above argument does not work. It should be possible to check this by using the Coulomb branch of $(\tilde{G}, \mathbf{N})$ in [Quiver, §2(ii)]. But we give an alternative argument in Lemma 6.9.

Further examples are given as toric hyper-Kähler manifolds. See §4(vii).

Let us also remark that this proposition is naturally predicted from the monopole formula (2.9), as was observed in [CHZ14, §5.1].

3(viii). Flavor symmetry group – deformation

Suppose that we have a larger group $\tilde{G}$ containing G as a normal subgroup. Let $G_F = \tilde{G}/G$. This is called the *flavor symmetry group* in physics literature. We suppose our G-module $\mathbf{N}$ extends to a $\tilde{G}$-module. We denote it by the same notation $\mathbf{N}$. The Hamiltonian reduction in 3(vii) above is an example when G_F is a torus. Note that $\mathcal{M}_C(G, \mathbf{N})$ in 3(vii) has a natural deformation and a family of quasi-projective varieties which are projective over $\mathcal{M}_C(G, \mathbf{N})$. The former is given by changing the level of the moment map, and the latter is given by considering GIT quotients for characters of $T_F^\vee$. We will give both constructions for arbitrary G_F. This property was expected in [Nak16, §5]. See original physics literature given there.

The deformation is easy, and is given here. Quasi-projective varieties will be given later in §3(ix).

Since G is a normal subgroup of $\tilde{G}$, the $G_\mathcal{O}$-action on Gr_G extends to $\tilde{G}_\mathcal{O}$. Moreover, as $\mathbf{N}$ is a representation of $\tilde{G}$, we have $\tilde{G}_\mathcal{O}$-actions on $\mathcal{T}$, $\mathcal{R}$, etc. Therefore we can consider the $\tilde{G}$-equivariant homology group $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R})$. It is a module over $H_{\tilde{G}}^*(\mathrm{pt})$ and has extra directions parametrized by $\mathrm{Spec}(H_{G_F}^*(\mathrm{pt}))$. We have the restriction homomorphism

$$H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}) \to H_*^{G_\mathcal{O}}(\mathcal{R}) = H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}) \otimes_{H_{G_F}^*(\mathrm{pt})} \mathbb{C}.$$

Proposition 3.22. *A convolution product $*$ defines an associative graded algebra structure on $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R})$. The restriction homomorphism $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}) \to H_*^{G_\mathcal{O}}(\mathcal{R})$ is an algebra homomorphism. The same is true for $\tilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times$, $G_\mathcal{O} \rtimes \mathbb{C}^\times$ equivariant homology groups.*

Applying Lemma 3.20 to the setting in the proof below, we see that $H^*_{G_F}(\text{pt})$ is central in $H^{\tilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times}_*(\mathcal{R})$. (Note that $\pi_1(\chi)$ is zero on $\mathcal{R}$.) Therefore $H^{\tilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times}_*(\mathcal{R}) \otimes_{H^*_{G_F}(\text{pt})} \mathbb{C}$ has an induced multiplication. Then the second assertion means that $H^{\tilde{G}_\mathcal{O} \rtimes \mathbb{C}^\times}_*(\mathcal{R}) \otimes_{H^*_{G_F}(\text{pt})} \mathbb{C} \cong H^{G_\mathcal{O} \rtimes \mathbb{C}^\times}_*(\mathcal{R})$ is an algebra isomorphism.

A proof of commutativity of the product on $H^{\tilde{G}_\mathcal{O}}_*(\mathcal{R})$ is the same as one for $H^{G_\mathcal{O}}_*(\mathcal{R})$, hence is postponed until §5.

Proof. Let us regard Gr_G as the moduli space of pairs $(\mathcal{P}, \varphi)$ as before. We have the induced $\tilde{G}$-bundle $\tilde{\mathcal{P}} \overset{\text{def.}}{=} \mathcal{P} \times_G \tilde{G}$ and its trivialization $\tilde{\varphi} \overset{\text{def.}}{=} \varphi \times_G \tilde{G} \colon \mathcal{P} \times_G \tilde{G} \to \tilde{G} \times D^*$. Moreover for the further induced G_F-bundle $\tilde{\mathcal{P}} \times_{\tilde{G}} G_F$, the trivialization $\tilde{\varphi} \times_{\tilde{G}} G_F$ extends across the origin $0 \in D$. Conversely a pair $(\tilde{\mathcal{P}}, \tilde{\varphi})$ such that the trivialization $\tilde{\varphi} \times_{\tilde{G}} G_F$ extends is coming from a pair $(\mathcal{P}, \varphi)$. Thus Gr_G can be regarded as the moduli space of such pairs $(\tilde{\mathcal{P}}, \tilde{\varphi})$.

Let $\tilde{G}^\mathcal{O}_\mathcal{K}$ be the inverse image of $(G_F)_\mathcal{O}$ under $\tilde{G}_\mathcal{K} \to (G_F)_\mathcal{K}$. The homomorphism $G_\mathcal{K} \to \tilde{G}^\mathcal{O}_\mathcal{K}$ induces a bijection $\text{Gr}_G = G_\mathcal{K}/G_\mathcal{O} \cong \tilde{G}^\mathcal{O}_\mathcal{K}/\tilde{G}_\mathcal{O}$, where the latter quotient $\tilde{G}^\mathcal{O}_\mathcal{K}/\tilde{G}_\mathcal{O}$ is compatible with the above scheme structure of Gr_G. As an analog of (2.3), we have

$$H^{\tilde{G}_\mathcal{O} \times \tilde{G}_\mathcal{O}}_*(\tilde{G}^\mathcal{O}_\mathcal{K}) \cong H^{\tilde{G}_\mathcal{O}}_*(\text{Gr}_G).$$

Let us modify (the lower row of) the diagram (3.2) as

(3.23) $\mathcal{T} \times \mathcal{R} \xleftarrow{\ p\ } \tilde{G}^\mathcal{O}_\mathcal{K} \times \mathcal{R} \xrightarrow{\ q\ } \tilde{G}^\mathcal{O}_\mathcal{K} \times_{\tilde{G}_\mathcal{O}} \mathcal{R} \xrightarrow{\ m\ } \mathcal{T},$

where maps are given by

$$\big([g_1, g_2 s], [g_2, s]\big) \leftarrow\!\shortmid (g_1, [g_2, s]) \mapsto \big[g_1, [g_2, s]\big] \mapsto [g_1 g_2, s].$$

It is exactly the same formula as (3.3) above, but we have used the description $\mathcal{R} = \{[g, s] \in \tilde{G}^\mathcal{O}_\mathcal{K} \times_{\tilde{G}_\mathcal{O}} \mathbf{N}_\mathcal{O} \mid gs \in \mathbf{N}_\mathcal{O}\}$, etc. The upper row of (3.2) is defined in the same way.

The same formula as in (3.4) gives actions

$$\tilde{G}_\mathcal{O} \times \tilde{G}_\mathcal{O} \curvearrowright \mathcal{T} \times \mathcal{R}, \qquad \tilde{G}_\mathcal{O} \times \tilde{G}_\mathcal{O} \curvearrowright \tilde{G}^\mathcal{O}_\mathcal{K} \times \mathcal{R},$$
$$\tilde{G}_\mathcal{O} \curvearrowright \tilde{G}^\mathcal{O}_\mathcal{K} \times_{\tilde{G}_\mathcal{O}} \mathcal{R}, \qquad \tilde{G}_\mathcal{O} \curvearrowright \mathcal{T}.$$

The above diagram (3.23) is equivariant. Also the diagram is given by morphisms of schemes as before.

We then define the convolution product $*$ on $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R})$ using (3.23) instead of (3.2).

The compatibility of two products on $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R})$ and $H_*^{G_\mathcal{O}}(\mathcal{R})$ follows from a commutative diagram connecting (3.2) to (3.23). The detail is omitted. $\square$

Remarks 3.24. (1) When G_F is a torus, we are in the setting §3(vii)3(vii). As we have remarked in the beginning of this subsection, the quotient $\mu_{T_F^\vee} \colon \mathcal{M}_C(\tilde{G}, \mathbf{N}) /\!\!/ T_F^\vee \to \mathfrak{t}_F$ gives a deformation of $\mathcal{M}_C(G, \mathbf{N})$ parametrized by $\mathfrak{t}_F$. Taking the quotient by $T_F^\vee$, but not imposing the moment map equation $\mu_{T_F} = 0$ means that we change the space from $\mathcal{R}_{\tilde{G}, \mathbf{N}}$ to $\mathcal{R}_{G, \mathbf{N}}$, but keep the group as $\tilde{G}_\mathcal{O}$. Therefore we have $\mathcal{M}_C(\tilde{G}, \mathbf{N}) /\!\!/ T_F^\vee = \operatorname{Spec} H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R})$ in this case.

In general, we take a maximal torus T_F of G_F, and set $\tilde{G}'$ as its inverse image in $\tilde{G}$. Then $\mathcal{M}_C(\tilde{G}', \mathbf{N}) /\!\!/ T_F^\vee = \operatorname{Spec} H_*^{\tilde{G}'_\mathcal{O}}(\mathcal{R})$ is a W_F-covering of $\operatorname{Spec} H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R})$, where W_F is the Weyl group of G_F. (Checking that $H_*^{\tilde{G}'_\mathcal{O}}(\mathcal{R}) \cong H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}) \otimes_{H_{\tilde{G}}^*(\mathrm{pt})} H_{\tilde{G}'}^*(\mathrm{pt})$ respects the multiplication is left as an exercise for the reader.)

(2) If $\tilde{G}$ acts on $\mathbf{N}$ through G, i.e., we have a group homomorphism $\rho \colon \tilde{G} \to G$ such that the $\tilde{G}$-action on $\mathbf{N}$ factors through ρ, we have $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}) \cong H_*^{G_\mathcal{O}}(\mathcal{R}) \otimes_{H_G^*(\mathrm{pt})} H_{\tilde{G}}^*(\mathrm{pt})$, where $H_G^*(\mathrm{pt}) \to H_{\tilde{G}}^*(\mathrm{pt})$ is given by ρ. Our deformation is trivial in this case. An example is the dilatation action of $\mathbb{C}^\times$ on $\mathbf{N}$. (See §6(ix).) Although $\tilde{G} = G \times \mathbb{C}^\times$ acts on $\mathbf{N}$, it factors through G in many occasions, say quiver gauge theories whose underlying graphs have no cycles.

3(ix). Flavor symmetry group – resolution

Let us continue the setting in the previous subsection. We have just constructed a deformation of $\mathcal{M}_C$ parametrized by $H_{G_F}^*(\mathrm{pt})$. Since $\mathcal{M}_C$ is supposed to be a hyper-Kähler manifold, a deformation and a (partial) resolution should come together. We shall construct the latter in this subsection. In fact, in the hyper-Kähler setting, one can construct simultaneous resolution of the deformation, after the base change to a W_F-cover. Therefore in view of Remark 3.24 we take a maximal torus T_F of G_F, and set $\tilde{G}'$ as its inverse image in $\tilde{G}$. Therefore we are in the setting of §3(vii)3(vii). We consider the GIT quotient of $\mathcal{M}_C(\tilde{G}', \mathbf{N})$ by $T_F^\vee$ with respect to a character λ_F of $T_F^\vee$. Let us denote it by $\mathcal{M}_C(\tilde{G}', \mathbf{N}) /\!\!/_{\lambda_F} T_F^\vee$. We have a natural projective

morphism $\mathcal{M}_C(\tilde{G}', \mathbf{N}) /\!\!/_{\lambda_F} T_F^\vee \to \mathcal{M}_C(\tilde{G}', \mathbf{N}) /\!\!/ T_F^\vee$, which fits in the diagram

$$
\begin{array}{ccc}
\operatorname{Spec} H_*^{\tilde{G}_{\mathcal{O}}}(\mathcal{R}) & \longrightarrow & \mathfrak{t}_F / W_F \\
\uparrow{\scriptstyle /W_F} & & \uparrow{\scriptstyle /W_F} \\
\end{array}
$$

$$
\begin{array}{ccccc}
\mathcal{M}_C(\tilde{G}', \mathbf{N}) /\!\!/_{\lambda_F} T_F^\vee & \longrightarrow & \mathcal{M}_C(\tilde{G}', \mathbf{N}) /\!\!/ T_F^\vee & \longrightarrow & \mathfrak{t}_F \\
\uparrow & & \uparrow & & \uparrow \\
\mu_{T_F^\vee}^{-1}(0) /\!\!/_{\lambda_F} T_F^\vee & \longrightarrow & \mu_{T_F^\vee}^{-1}(0) /\!\!/ T_F^\vee \overset{\S 3(\mathrm{vii})}{=} \mathcal{M}_C(G, \mathbf{N}) & \longrightarrow & \{0\}.
\end{array}
$$

This is reminiscent of the Grothendieck-Springer resolution of $\mathfrak{g}^*$ and the corresponding diagram for quiver varieties.

Let us give a description of $\mathcal{M}_C(\tilde{G}', \mathbf{N}) /\!\!/_{\lambda_F} T_F^\vee$ in terms of a homology group of a modification of $\mathcal{R}$.

In order to simplify the notation, let us replace $\tilde{G}$ (resp. G_F) by $\tilde{G}'$ (resp. T_F), and assume $\tilde{G} = \tilde{G}'$, $G_F = T_F$. We also denote $\mathcal{T}_{\tilde{G}, \mathbf{N}}$, $\mathcal{R}_{\tilde{G}, \mathbf{N}}$ by $\tilde{\mathcal{T}}$, $\tilde{\mathcal{R}}$ respectively for short.

A $\tilde{G}$-bundle $\mathcal{P}$ induces a T_F-bundle by $\mathcal{P} \times_{\tilde{G}} T_F$. This gives a morphism $\operatorname{Gr}_{\tilde{G}} \to \operatorname{Gr}_{T_F}$. Composing it with $\tilde{\mathcal{T}} \to \operatorname{Gr}_{\tilde{G}}$ or $\tilde{\mathcal{R}} \to \operatorname{Gr}_{\tilde{G}}$, we have

$$\tilde{\pi}\colon \tilde{\mathcal{T}} \text{ or } \tilde{\mathcal{R}} \to \operatorname{Gr}_{T_F}.$$

Let $\tilde{\mathcal{T}}^{\lambda_F}$ or $\tilde{\mathcal{R}}^{\lambda_F} = \tilde{\pi}^{-1}(\lambda_F)$ be a fiber of this projection at a coweight λ_F of G_F. It is preserved under the action of $\tilde{G}_{\mathcal{O}}$. We have $H_*^{\tilde{G}_{\mathcal{O}}}(\tilde{\mathcal{R}}) = \bigoplus_{\lambda_F} H_*^{\tilde{G}_{\mathcal{O}}}(\tilde{\mathcal{R}}^{\lambda_F})$ by §3(v). It corresponds to $T_F^\vee$-action on $\mathcal{M}_C(\tilde{G}, \mathbf{N})$.

Proposition 3.25. *Consider the $\mathbb{Z}_{\geq 0}$-graded algebra $\bigoplus_{n \geq 0} H_*^{\tilde{G}_{\mathcal{O}}}(\tilde{\mathcal{R}}^{n\lambda_F})$. We have*

$$\mathcal{M}_C(\tilde{G}, \mathbf{N}) /\!\!/_{\lambda_F} T_F^\vee \cong \operatorname{Proj}(\bigoplus_{n \geq 0} H_*^{\tilde{G}_{\mathcal{O}}}(\tilde{\mathcal{R}}^{n\lambda_F})).$$

Similarly we have

$$\mu_{T_F^\vee}^{-1}(0) /\!\!/_{\lambda_F} T_F^\vee \cong \operatorname{Proj}(\bigoplus_{n \geq 0} H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}^{n\lambda_F})).$$

This is clear from the definition of the left hand side. It is Proj of the ring of $T_F^\vee$-semi-invariants with respect to the character $\lambda_F\colon T_F^\vee \to \mathbb{C}^\times$. The semi-invariants ring is the graded algebra in question.

For the second isomorphism, note that $H_*^{G_{\mathcal{O}}}(\tilde{\mathcal{R}}) = H_*^{\tilde{G}_{\mathcal{O}}}(\tilde{\mathcal{R}}) \otimes_{H_{T_F}^*(\mathrm{pt})} \mathbb{C}$ has the induced multiplication. This is obvious if we use the commutativity,

which will be proved later. But it is true even without commutativity as we have already know $H^*_{G_F}(\mathrm{pt})$ is central, as remarked after Proposition 3.22.

Let us denote $\mathrm{Proj}(\bigoplus_{n \geq 0} H^{G_\mathcal{O}}_*(\tilde{\mathcal{R}}^{n\lambda_F}))$ by $\mathcal{M}^{\lambda_F}_C(G, \mathbf{N})$. We have a projective morphism $\boldsymbol{\pi} \colon \mathcal{M}^{\lambda_F}_C(G, \mathbf{N}) \to \mathcal{M}_C(G, \mathbf{N})$ as $H^{G_\mathcal{O}}_*(\tilde{\mathcal{R}}^0) = H^{G_\mathcal{O}}_*(\mathcal{R}) = \mathbb{C}[\mathcal{M}_C(G, \mathbf{N})]$. Since $\bigoplus_{n \geq 0} H^{G_\mathcal{O}}_*(\tilde{\mathcal{R}}^{n\lambda_F})$ is finitely generated, we can take sufficiently large n_0 so that $\bigoplus_{n \geq 0} H^{G_\mathcal{O}}_*(\tilde{\mathcal{R}}^{nn_0\lambda_F})$ is generated by $H^{G_\mathcal{O}}_*(\tilde{\mathcal{R}}^{n_0\lambda_F})$ as a $\mathbb{C}[\mathcal{M}_C(G, \mathbf{N})]$-algebra. Therefore we have the corresponding line bundle over $\mathcal{M}^{\lambda_F}_C(G, \mathbf{N})$. Let $\mathcal{L}_{\lambda_F}$ denote its $1/n_0$-th power, considered as a $\mathbb{Q}$-line bundle.

Remark 3.26. The dimension of $H^{G_\mathcal{O}}_*(\tilde{\mathcal{R}}^{n\lambda_F}))$ is given by a formula similar to Proposition 2.7 and (2.9). In fact, we take the stratification $\tilde{\mathcal{R}} = \bigsqcup \tilde{\mathcal{R}}_{\tilde{\lambda}}$ parametrized by dominant coweights $\tilde{\lambda}$ of $\tilde{G}$. Then $\tilde{\mathcal{R}}^{\lambda_F}$ is the union of strata $\tilde{\mathcal{R}}_{\tilde{\lambda}}$ such that $\tilde{\lambda}$ is sent to λ_F under $\tilde{G} \to T_F$. Such an extension of the monopole formula was given in [CHMZ14]. See [Nak16, §5(i)] for a review.

3(x). Previously known examples

Let us identify previous constructions in the literature as special cases of our Coulomb branches.

(a). Pure gauge theories. Consider $\mathbf{N} = 0$. Then $\mathcal{R} = \mathrm{Gr}_G$. The convolution algebra $H^{G_\mathcal{O}}_*(\mathrm{Gr}_G)$ was calculated in [BFM05, Th. 2.12], and was attributed to an earlier work by Peterson [Pet97, Kos96]. It is the algebraic variety $\mathfrak{Z}^{G^\vee}_{\mathfrak{g}^\vee}$ formed by the pairs (g, x) such that x lies in a (fixed) Kostant slice in $\mathfrak{g}^\vee$, and $g \in G^\vee$ satisfies $\mathrm{Ad}_g(x) = x$. Combining with [Bie97], we will prove that the Coulomb branch $\mathcal{M}_C$ is the moduli space of solutions of Nahm's equations for the Langlands dual group $G^\vee_c$ in Theorem A.1. For $G = \mathrm{SL}(k)$, it recovers results proved by physical arguments ([SW97] for $k = 2$, [CH97] for arbitrary k). See §A for detail.

(b). Adjoint matters. Consider $\mathbf{N} = \mathfrak{g}$, the adjoint representation of G. We consider the dilatation action on $\mathbf{N}$ as a flavor symmetry group ($\tilde{G} = G \times \mathbb{C}^\times$, $G_F = \mathbb{C}^\times$). The space $\mathcal{R}$ in this case is

$$\mathcal{R} = \{(\xi, [g]) \in \mathfrak{g}_\mathcal{O} \times \mathrm{Gr}_G \mid \mathrm{Ad}_{g^{-1}}(\xi) \in \mathfrak{g}_\mathcal{O}\},$$

which is a variant of the affine Grassmannian Steinberg variety, denoted by Λ in [BFM05, §7]. (In fact, Λ is a closed subvariety of $\mathcal{R}$, and the inclusion induces $H^{G_\mathcal{O}}_*(\Lambda) \cong H^{G_\mathcal{O}}_*(\mathcal{R})$.) In [BFM05, §7], it was shown that $K^{G_\mathcal{O}}(\Lambda)$

is isomorphic to $\mathbb{C}[T^\vee \times T]^W$. The argument for K-theory in [BFM05] uses its specific features (certain coherent sheaves obtained as associated graded of certain D-modules). The homology case can be deduced from the K-theory case via Chern character homomorphism as we will see in [Quiver, Proposition 3.23]. But we also give another proof in Proposition 6.14. Recall that it was shown that the equivariant K-group of the affine flag variety analog of Λ is isomorphic to the Cherednik double affine Hecke algebra in [VV10]. (See also an earlier work [Vas05].) Therefore, when we include the loop rotation $\mathbb{C}^\times$ and the flavor symmetry group $G_F = \mathbb{C}^\times$, the equivariant homology group $H_*^{\tilde{G}_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R})$ is expected to be the spherical subalgebra of the graded Cherednik algebra (alias the trigonometric degeneration of the double affine Hecke algebra). [3] In fact, representations of the whole graded Cherednik algebra were constructed on equivariant homology groups of affine Springer fibers [OY16]. Therefore we believe that the only remaining task is a matter of checking.

4. The abelian case

In this section we determine the Coulomb branch and its quantization when the group is a torus. We obtain an explicit presentation of the ring $H_*^{T_{\mathcal{O}}}(\mathcal{R})$. This presentation is the same as one proposed in [BDG15, §3] by a physical intuition.

4(i). The main result - the non-quantized case

Let T be a torus and let Y denote its coweight lattice. We denote by $\mathfrak{t}$ the Lie algebra of T and by $\mathfrak{t}^*$ its dual space. Let $\mathbf{N}$ be a representation of T given by a bunch of characters $\xi_1, \ldots, \xi_n$. Note that each ξ_i can be viewed as an element of $\mathfrak{t}^*$.

For two integers k, l let us set

$$d(k,l) = \begin{cases} 0 & \text{if } k \text{ and } l \text{ have the same sign,} \\ \min(|k|, |l|) & \text{if } k \text{ and } l \text{ have different signs.} \end{cases}$$

Theorem 4.1. *The T-equivariant Borel-Moore homology of $\mathcal{R} \equiv \mathcal{R}_{T,\mathbf{N}}$ is generated by the algebra $\mathrm{Sym}(\mathfrak{t}^*)$ together with symbols r^λ, $\lambda \in Y$ subject to*

---

[3] After this paper was written, it is proved in [KN18] in type A.

the following relation:

$$(4.2) \qquad r^\lambda r^\mu = \prod_{i=1}^{n} \xi_i^{d(\xi_i(\lambda), \xi_i(\mu))} r^{\lambda+\mu}.$$

Remark 4.3. It is easy to see that the above algebra is graded if the degree of any $y \in \mathfrak{t}^*$ is 2 and we also set

$$\deg r^\lambda = \sum_{i=1}^{n} |\xi_i(\lambda)|.$$

This is the grading given by Δ in (2.10).

Proof. First, the algebra $H_*^T(\mathcal{R})$ clearly contains $H_T^*(\mathrm{pt}) = \mathrm{Sym}(\mathfrak{t}^*)$. The multiplication is $H_T^*(\mathrm{pt})$-linear in the first variable by definition. It is also in the second variable by Lemma 3.20. On the other hand, the affine Grassmannian of T has connected components parametrized by Y. Each such components consists of one point (we ignore nilpotents here). The space $\mathcal{T}$ is identified with the disjoint union $\bigsqcup_{\lambda \in Y} \{\lambda\} \times z^\lambda \mathbf{N}_\mathcal{O}$ via the embedding $\mathcal{T} \subset \mathrm{Gr}_T \times \mathbf{N}_\mathcal{K}$. We also have $\mathcal{R} = \bigsqcup_{\lambda \in Y} \{\lambda\} \times (z^\lambda \mathbf{N}_\mathcal{O} \cap \mathbf{N}_\mathcal{O})$. We denote by r^λ the fundamental class of the component of $\mathcal{R}$, corresponding to λ. Clearly, as a vector space we have

$$H_*^{T_\mathcal{O}}(\mathcal{R}) = \bigoplus_{\lambda \in Y} \mathrm{Sym}(\mathfrak{t}^*) r^\lambda$$

It remains to show that (4.2) holds. For this let us describe the part of the convolution diagram, corresponding to the convolution of the components, corresponding to λ and μ.

We use the identification

$$T_\mathcal{K} \times_{T_\mathcal{O}} \mathcal{R} \cong \bigsqcup_{\lambda, \nu \in Y} \{\lambda\} \times \{\nu\} \times (z^\nu \mathbf{N}_\mathcal{O} \cap z^\lambda \mathbf{N}_\mathcal{O})$$

given by $[g_1, [g_2, s]] \mapsto ([g_1], [g_1 g_2], g_1 g_2 s)$. We define $p' : T_\mathcal{K} \times_{T_\mathcal{O}} \mathcal{R} \to \mathcal{T} \times \mathcal{R}$ by

$$(4.4) \quad \{\lambda\} \times \{\nu\} \times (z^\nu \mathbf{N}_\mathcal{O} \cap z^\lambda \mathbf{N}_\mathcal{O}) \ni (\lambda, \nu, s)$$
$$\longmapsto (\lambda, s, \nu - \lambda, z^{-\lambda} s) \in \{\lambda\} \times z^\lambda \mathbf{N}_\mathcal{O} \times \{\nu - \lambda\} \times (z^{\nu-\lambda} \mathbf{N}_\mathcal{O} \cap \mathbf{N}_\mathcal{O}).$$

Let us decompose $p' = (p'_\mathcal{T}, p'_\mathcal{R})$ as before. We have $p'_\mathcal{T} \circ q = p_\mathcal{T}$. For $\mathcal{R}$, we define $a : T_\mathcal{K} \times \mathcal{R} \to T_\mathcal{K} \times \mathcal{R}$ by $(g_1, [g_2, s]) \mapsto (g_1, [z^\lambda g_1^{-1} g_2, s])$ when the

class of g_1 is λ. (Note $z^\lambda g_1^{-1} \in T_{\mathcal{O}}$.) Then $p'_{\mathcal{R}} \circ q \circ a = p_{\mathcal{R}}$. Therefore

$$(4.5) \qquad (q^*)^{-1} p_{\mathcal{R}}^* \boldsymbol{\omega}_{\mathcal{R}} = p_{\mathcal{R}}'^* \boldsymbol{\omega}_{\mathcal{R}}, \qquad (q^*)^{-1} p_{\mathcal{R}}^* \boldsymbol{\omega}_{\mathcal{R}} = (q^*)^{-1} a^* q^* p_{\mathcal{T}}'^* \boldsymbol{\omega}_{\mathcal{T}}.$$

In the abelian case, $\mathcal{T} \times \mathcal{R}$, $T_{\mathcal{K}} \times_{T_{\mathcal{O}}} \mathcal{R}$ (resp. $T_{\mathcal{K}} \times \mathcal{R}$) are unions of vector spaces (resp. products of groups and vector spaces), in particular they are smooth and hence their dualizing sheaves are isomorphic to constant sheaves up to shifts by the Poincaré duality. Therefore $p_{\mathcal{R}}'^* \boldsymbol{\omega}_{\mathcal{R}} \cong \mathbb{C}_{T_{\mathcal{K}} \times_{T_{\mathcal{O}}} \mathcal{R}}$, $p_{\mathcal{T}}'^* \boldsymbol{\omega}_{\mathcal{T}} \cong \mathbb{C}_{T_{\mathcal{K}} \times_{T_{\mathcal{O}}} \mathcal{R}}$ up to shifts. Looking at the construction of isomorphisms in (3.6), we find that this Poincaré duality and (3.6) are the same under (4.5). We further replace p' by multiplying z^λ, composing the inclusion $z^\nu \mathbf{N}_{\mathcal{O}} \cap z^\lambda \mathbf{N}_{\mathcal{O}} \subset z^\nu \mathbf{N}_{\mathcal{O}}$ in the second factor, and dropping unnecessary entries λ, ν:

$$p' : z^\lambda \mathbf{N}_{\mathcal{O}} \cap z^\nu \mathbf{N}_{\mathcal{O}} \ni s \mapsto (s, s) \in z^\lambda \mathbf{N}_{\mathcal{O}} \times z^\nu \mathbf{N}_{\mathcal{O}}.$$

Then the morphism m is given by the projection to the second factor $z^\nu \mathbf{N}_{\mathcal{O}}$. As $(q^*)^{-1} a^* q^*$ in (4.5) does not affect the computation, we have $r^\lambda r^\mu = m_* p'^* [(z^\lambda \mathbf{N}_{\mathcal{O}} \cap \mathbf{N}_{\mathcal{O}}) \times z^\nu \mathbf{N}_{\mathcal{O}}]$.

To simplify the notation, let us give a proof in the case $n = 1$ (the general case is essentially a word-by-word repetition). We set $\xi = \xi_1$. Also, set $a = \xi(\lambda)$, $b = \xi(\mu)$. In the above notation we use $\nu = \lambda + \mu$, so $a + b = \xi(\nu)$.

From the above argument, the convolution product $r^\lambda r^\mu$ is given as follows. First, consider the vector space $W = z^a \mathcal{O} \oplus z^{a+b} \mathcal{O}$. It contains subspaces $V_{12} = \mathcal{O} \cap z^a \mathcal{O} \oplus z^{a+b} \mathcal{O}$, $V_{23} = z^a \mathcal{O} \cap z^{a+b} \mathcal{O}$ (where V_{23} is embedded into W by means of the diagonal embedding). Let also m denote the projection from W to $z^{a+b} \mathcal{O}$. We denote by V the subspace of $z^{a+b} \mathcal{O}$ equal to $\mathcal{O} \cap z^{a+b} \mathcal{O}$. Clearly, m sends $V_{12} \cap V_{23} = \mathcal{O} \cap z^a \mathcal{O} \cap z^{a+b} \mathcal{O}$ to V (and when restricted to $V_{12} \cap V_{23}$ the map p is injective).

Let x_{12} and x_{23} be the fundamental classes of V_{12} and V_{23} viewed as elements of the T-equivariant Borel-Moore homology of W. There exists a class x in $H_*^T(V)$ such that $m_*(x_{12} \cap x_{23})$ is equal to the direct image of x from V to $z^{a+b} \mathcal{O}$. Then x is the product of $r^\lambda r^\mu$.

On the other hand, let L be any vector space with a T-action and let L_1, L_2 be two T-invariant subspaces. Let x_{L_i} (resp. $x_{L_1 \cap L_2}$) be the fundamental class of L_i (resp. $L_1 \cap L_2$) viewed as an element of $H_*^T(L)$. Then $x_{L_1} \cap x_{L_2} = x_{L_1 \cap L_2} \cdot e(\mathrm{Coker}(L/L_1 \cap L_2 \to L/L_1 \oplus L/L_2))$, where $e(\ )$ denotes the equivariant Euler class. Hence in our case we see that $r^\lambda r^\mu$ equals $r^{\lambda+\mu}$ multiplied by

$$(4.6) \qquad e(\mathrm{Coker}(W/V_{12} \cap V_{23} \to W/V_{12} \oplus W/V_{23})) \cdot e(V/V_{12} \cap V_{23}).$$

Let us now look at the different options for signs of a and b.

1) $a \geq 0, b \geq 0$.

In this case $V_{12} = z^a \mathcal{O} \oplus z^{a+b} \mathcal{O}$, $V_{23} = z^{a+b} \mathcal{O} = V_{12} \cap V_{23} = V$. Hence the 2nd equivariant Euler class in (4.6) is equal to 1. Also $W/V_{12} \cap V_{23} = z^a \mathcal{O}$, $W/V_{12} = 0$, $W/V_{23} = z^a \mathcal{O}$, hence the 1st equivariant Euler class is equal to 1 as well.

2) $a \leq 0, b \leq 0$.

In this case $V_{12} = \mathcal{O} \oplus z^{a+b} \mathcal{O}$, $V_{23} = z^a \mathcal{O}$, $V = V_{12} \cap V_{23} = \mathcal{O}$. Hence the 2nd equivariant Euler class is equal to 1 again. Also $W/V_{12} = z^a \mathcal{O}/\mathcal{O}$, $W/V_{23} = z^{a+b} \mathcal{O}$ and $W/V_{12} \cap V_{23} = z^a \mathcal{O}/\mathcal{O} \oplus z^{a+b} \mathcal{O}$. Hence the 1st equivariant Euler class is again 1.

3) $a \geq 0, b \leq 0$.

In this case we have $V_{12} = z^a \mathcal{O} \oplus z^{a+b} \mathcal{O}$, $V_{23} = z^a \mathcal{O}$, $V_{12} \cap V_{23} = z^a \mathcal{O}$, $V = \mathcal{O} \cap z^{a+b} \mathcal{O}$. So $W/V_{12} = 0$, $W/V_{23} = z^{a+b} \mathcal{O}$, $W/V_{12} \cap V_{23} = z^{a+b} \mathcal{O}$ and hence the first equivariant Euler class in (4.6) is equal to 1. On the other hand the 2nd equivariant Euler class is equal to $e(\mathcal{O} \cap z^{a+b} \mathcal{O}/z^a \mathcal{O})$ which is equal to ξ^{-b} if $a + b \geq 0$ and ξ^a if $a + b \leq 0$. This is exactly $\xi^{d(a,b)}$.

4) $a \leq 0, b \geq 0$.

In this case we have $V_{12} = \mathcal{O} \oplus z^{a+b} \mathcal{O}$, $V_{23} = z^{a+b} \mathcal{O}$, $V_{12} \cap V_{23} = \mathcal{O} \cap z^{a+b} \mathcal{O} = V$. Hence in this case the 2nd equivariant Euler class (4.6) is equal to 1. On the other hand, assume that $a + b \geq 0$. Then $V_{12} \cap V_{23} = z^{a+b} \mathcal{O}$ and $W/V_{12} \cap V_{23} = z^a \mathcal{O}$. On the other hand, $W/V_{12} = z^a \mathcal{O}/\mathcal{O}$, $W/V_{23} = z^a \mathcal{O}$ and hence the first equivariant Euler class in (4.6) is equal to $e(z^a \mathcal{O}/\mathcal{O}) = \xi^{|a|}$. But since $a + b \geq 0$ we have $|a| = d(a, b)$. Similarly, if $a + b \leq 0$, then $V_{12} \cap V_{23} = \mathcal{O}$. Hence $W/V_{12} = z^a \mathcal{O}/\mathcal{O}$, $W/V_{23} = z^a \mathcal{O}$ and $W/V_{12} \cap V_{23} = z^a \mathcal{O} \oplus z^{a+b} \mathcal{O}/\mathcal{O}$. Hence the equivariant Euler class in question is equal to $e(z^a \mathcal{O}/z^{a+b} \mathcal{O}) = \xi^b = \xi^{d(a,b)}$. $\square$

4(ii). The quantized version

We now want to do everything $\mathbb{C}^\times$-equivariantly (where $\mathbb{C}^\times$ acts by loop rotation). Let $\hbar$ denote the generator of $H^*_{\mathbb{C}^\times}(\mathrm{pt})$. Then the description of $H^{T_\mathcal{O} \rtimes \mathbb{C}^\times}_*(\mathcal{R})$ is similar, except the relation (4.2) now reads

$$(4.7) \qquad\qquad r^\lambda r^\mu = \prod_{i=1}^n A_i(\lambda, \mu) \, r^{\lambda+\mu},$$

where

$$A_i(\lambda, \mu) = \begin{cases} \prod_{j=1}^{d(\xi_i(\lambda), \xi_i(\mu))} (\xi_i + (\xi_i(\lambda) - j + \tfrac{1}{2})\hbar) & \text{if } \xi_i(\lambda) \geq 0 \geq \xi_i(\mu), \\ \prod_{j=1}^{d(\xi_i(\lambda), \xi_i(\mu))} (\xi_i + (\xi_i(\lambda) + j - \tfrac{1}{2})\hbar) & \text{if } \xi_i(\lambda) \leq 0 \leq \xi_i(\mu), \\ 1 & \text{otherwise.} \end{cases}$$

The proof easily follows from the same calculations as above. Note that $1/2$ is coming from the action of $\mathbb{C}^\times$ on $\mathbf{N}$ with weight $1/2$.

In addition, $\mathrm{Sym}_{\mathbb{C}[\hbar]}(\mathfrak{t}^*)$ now does not commute with r^λ's. But it is easy to see that for any $\alpha \in \mathfrak{t}^*$ we have

$$(4.8) \qquad\qquad [r^\lambda, \alpha] = \hbar\alpha(\lambda)r^\lambda.$$

See Lemma 3.20.

4(iii). Flavor symmetries

Let $\widetilde{T} = T \times (\mathbb{C}^\times)^n$. Then $\widetilde{T}$ acts naturally on $\mathbf{N}$ and thus also on $\mathcal{R}$ and we can consider the algebra $H_*^{\widetilde{T}_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R})$. This algebra has a presentation similar to the above, except that the relation (4.7) now reads

$$(4.9) \qquad\qquad r^\lambda r^\mu = \prod_{i=1}^n \widetilde{A}_i(\lambda, \mu)\, r^{\lambda + \mu},$$

where

$$\widetilde{A}_i(\lambda, \mu) = \begin{cases} \prod_{j=1}^{d(\xi_i(\lambda), \xi_i(\mu))} (b_i + \xi_i + (\xi_i(\lambda) - j + \tfrac{1}{2})\hbar) & \text{if } \xi_i(\lambda) \geq 0 \geq \xi_i(\mu), \\ \prod_{j=1}^{d(\xi_i(\lambda), \xi_i(\mu))} (b_i + \xi_i + (\xi_i(\lambda) + j - \tfrac{1}{2})\hbar) & \text{if } \xi_i(\lambda) \leq 0 \leq \xi_i(\mu), \\ 1 & \text{otherwise,} \end{cases}$$

where $b_1, \ldots, b_n$ denote the equivariant parameters for the torus $(\mathbb{C}^\times)^n$.

4(iv). Examples

Let $T = \mathbb{C}^\times, n = 1$ and take ξ to be the N-th power the standard character. In this case $\Lambda = \mathbb{Z}$. Set $x = r^1, y = r^{-1}$ (note that r^{-1} is not the inverse

of r^1!). Let also a denote the generator of $\mathfrak{t}^*$. Then x, y and a generate $H_*^{T\circ\times\mathbb{C}^\times}(\mathcal{R})$ and we have the following relations:

$$xy = \prod_{j=1}^{N}\left(a + \left(j - \frac{1}{2}\right)\hbar\right), \quad yx = \prod_{j=1}^{N}\left(a - \left(j - \frac{1}{2}\right)\hbar\right),$$

$$[x, a] = \hbar x, \quad [y, a] = -\hbar y$$

When $\hbar = 0$ we get a commutative ring, generated by x, y, a with relation $xy = a^N$.

On the other hand, let $\mathbf{N}$ be the direct sum of N standard characters. We define x, y and a in the same way as above. Then we have

$$xy = \left(a + \frac{1}{2}\hbar\right)^N, \quad yx = \left(a - \frac{1}{2}\hbar\right)^N, \quad [x, a] = \hbar x, \quad [y, a] = -\hbar y.$$

Note that for $\hbar = 0$ we get the same algebra as before.

Let $\deg_h$ denote the half of the homological degree. We have $\deg_h x = 0$, $\deg_h y = N$, $\deg_h a = 1$. On the other hand, the degree $\deg$ given by $\Delta(\lambda)$ by (2.10) is $\deg x = N/2$, $\deg y = N/2$, $\deg a = 1$. If we multiply it by 2, it is the weight of the $\mathbb{C}^\times$-action, which is the restriction of the SU(2)-action on $\{xy = a^N\} = \mathbb{C}^2/(\mathbb{Z}/N\mathbb{Z})$, rotating the hyper-Kähler structure. Namely we identify $\mathbb{C}^2$ with the quaternions $\mathbb{H} = \{z_1 + z_2 j \mid z_1, z_2 \in \mathbb{C}\}$. The hyper-Kähler structure is given by multiplication of i, j, k from the left, with the standard inner product. The action of $\mathbb{Z}/N\mathbb{Z}$ is given by multiplication of Nth roots of unity from the right. Then $x = z_1^N$, $y = z_2^N$, $a = z_1 z_2$. The action of SU(2) = Sp(1) is the multiplication of unit quaternions from the left. If we restrict to U(1), unit complex numbers, z_1, z_2 both have weight 1.

4(v). Changing ξ to $-\xi$

Let us assume that we change one of the ξ to $-\xi$; let us denote by $\mathbf{N}_i$ the corresponding new representation of T. We claim that the resulting algebra $\mathcal{A}(T, \mathbf{N}_i)$ is isomorphic to $\mathcal{A}(T, \mathbf{N})$. Then define a map $\sigma\colon \mathcal{A}(T, \mathbf{N}_i) \to \mathcal{A}(T, \mathbf{N})$ which is equal to identity on $\mathfrak{t}^*$ and such that

$$\sigma(r^\lambda) = \begin{cases} r^\lambda & \text{if } \xi_i(\lambda) \le 0 \\ (-1)^{\xi_i(\lambda)} r^\lambda & \text{if } \xi_i(\lambda) > 0 \end{cases}$$

In particular, we see that the algebra $\mathcal{A}(T, \mathbf{N})$ depends only on $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$ and not on $\mathbf{N}$.

The same is true for the quantized case. (It is not true if we do not introduce the weight $1/2$ action of $\mathbb{C}^\times$ on $\mathbf{N}$.)

4(vi). Changing the representation

Let $\mathbf{V}$ be another representation. Then we claim that there is a natural embedding $\mathcal{A}(T, \mathbf{N} \oplus \mathbf{V}) \hookrightarrow \mathcal{A}(T, \mathbf{N})$. To construct it, it is enough to assume that $\mathbf{V} = \xi$ for some ξ. Then the η which is equal to identity of $\mathrm{Sym}(\mathfrak{t}^*)$ and such that

(4.10)
$$\eta(r^\lambda) = \begin{cases} \xi^{-\xi(\lambda)} r^\lambda & \text{if } \xi(\lambda) < 0 \\ r^\lambda & \text{otherwise} \end{cases}$$

defines the embedding.

This embedding is given by the pull-back homomorphism $H_*^{T_\mathcal{O}}(\mathcal{R}_{T, \mathbf{N} \oplus \mathbf{V}}) \to H_*^{T_\mathcal{O}}(\mathcal{R}_{T, \mathbf{N}})$ with respect to the embedding $\mathcal{T}_{T, \mathbf{N}} \subset \mathcal{T}_{T, \mathbf{N} \oplus \mathbf{V}}$ of a subbundle. In §5(iv), we consider a similar pull-back with respect to $\mathbf{z} \colon \mathrm{Gr}_G \hookrightarrow \mathcal{T}_{G, \mathbf{N}}$ for general $(G, \mathbf{N})$. The formula (4.10) is understood as the multiplication of $e(z^\lambda \mathbf{V}_\mathcal{O} / z^\lambda \mathbf{V}_\mathcal{O} \cap \mathbf{V}_\mathcal{O}) = e(z^{\xi(\lambda)} \mathcal{O} / z^{\max(\xi(\lambda), 0)} \mathcal{O})$.

In the quantized case, we define

$$\eta(r^\lambda) = \begin{cases} \displaystyle\prod_{j=0}^{-\xi(\lambda)-1} (\xi + (\xi(\lambda) + j + \frac{1}{2}) \hbar) r^\lambda & \text{if } \xi(\lambda) < 0, \\ r^\lambda & \text{otherwise.} \end{cases}$$

Using (4.8), we can check that this is an algebra homomorphism.

The following observation might be important: although for general $\mathbf{V}$ the above map does not respect the gradings on $\mathcal{A}(T, \mathbf{N} \oplus \mathbf{V})$ and $\mathcal{A}(T, \mathbf{N})$, the gradings *are* preserved if $\mathbf{V}$ is self-dual.

Remark 4.11. We can recover Theorem 4.1 and its quantized version (4.7) from the above formulas together with $r^\lambda r^\mu = r^{\lambda+\mu}$ when $\mathbf{N} = 0$.

4(vii). Toric hyper-Kähler manifolds

Consider a short exact sequence

$$0 \to \mathbb{Z}^{d-n} \xrightarrow{\alpha} \mathbb{Z}^d \xrightarrow{\beta} \mathbb{Z}^n \to 0,$$

and the associated sequence

$$(4.12) \qquad 1 \to G = T^{d-n} \xrightarrow{\alpha} T^d \xrightarrow{\beta} G_F = T^n \to 1.$$

Let $\mathbf{N} = \mathbb{C}^d$, considered as a representation of G through α. By §3(vii)3(vii), the Coulomb branch $\mathcal{M}_C(G, \mathbf{N})$ is the Hamiltonian reduction of $\mathcal{M}_C(T^d, \mathbb{C}^d)$ by $G_F^\vee$. We have $\mathcal{M}_C(T^d, \mathbb{C}^d) \cong \mathbb{C}^{2d}$ by Theorem 4.1. Hence $\mathcal{M}_C(G, \mathbf{N})$ is, by definition, a toric hyper-Kähler manifold associated with the dual exact sequence of (4.12), introduced by Bielawski and Dancer in [BD00].[4]

The coordinate ring of the Hamiltonian reduction of $\mathbb{C}^{2d}$ by $G_F^\vee$ has the presentation given by Theorem 4.1. In fact, this can be checked directly. See [Nak16, §5(ii)].

5. Codimension 1 reduction

Recall that T is a maximal torus of G with Lie algebra $\mathfrak{t}$. Let W be the Weyl group of G.

In this section, we develop a tool to analyze $\mathcal{M}_C$ based on the idea in [BFM05]. It says that it is enough to find an affine scheme $\mathcal{M}$, flat over $\mathfrak{t}/W$ such that it is equal to $\mathcal{M}_C$ over an open subset $\mathfrak{t}^\bullet/W$ of $\mathfrak{t}/W = \mathrm{Spec}(H_G^*(\mathrm{pt}))$ whose complement has codimension at least 2. Then automatically $\mathcal{M}_C = \mathcal{M}$. See Theorem 5.26 for more detail.

It is easy to determine $\mathcal{M}_C$ on a smaller open subset $\mathfrak{t}^\circ/W$ of $\mathfrak{t}/W$, the complement of certain hyperplanes. We prove $H_*^{G^\circ}(\mathcal{R})|_{\mathfrak{t}^\circ/W} \cong \mathbb{C}[\mathfrak{t} \times T^\vee]|_{\mathfrak{t}^\circ}^W$, where $T^\vee$ is the dual torus of T. This will be done in §5(v) after preparation. At a point t in $(\mathfrak{t}^\bullet \setminus \mathfrak{t}^\circ)/W$, we will show that $\mathcal{M}_C$ is the Coulomb branch of another pair $(G', \mathbf{N}')$ such that G' has semisimple rank at most 1. (See Lemma 5.1.) We can determine $\mathcal{M}_C(G', \mathbf{N}')$: §4 for abelian, Lemma 6.9 for $G' = \mathrm{PGL}(2)$ or $\mathrm{SL}(2)$, and the proof of Proposition 6.12 in general. Therefore we just need to look for $\mathcal{M}$ such that it is isomorphic to $\mathcal{M}_C(G', \mathbf{N}')$ for each $t \in (\mathfrak{t}^\bullet \setminus \mathfrak{t}^\circ)/W$. Using these, we can determine, for example, $(G, \mathbf{N})$ associated with a quiver gauge theory of type ADE. See [Quiver, Theorem 3.1].

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[4]The name 'toric hyper-Kähler manifold' and a special class of examples were introduced earlier in [Got92]. Many people use a new name 'hypertoric manifold', but we use the original name to respect the contribution of Goto, Bielawski-Dancer.

5(i). Fixed points and generalized roots

Lemma 5.1. *Let $t \in \operatorname{Lie} T$. Let $\mathcal{R}^t_{G,\mathbf{N}}$ be the fixed point set of $\exp(\mathbb{R}t)$ in $\mathcal{R}_{G,\mathbf{N}}$, i.e., the zero locus of the vector field generated by t. Then*

$$\mathcal{R}^t_{G,\mathbf{N}} \cong \mathcal{R}_{Z_G(t),\mathbf{N}^t},$$

where $Z_G(t)$ is the centralizer of t in G, and $\mathbf{N}^t$ is the subspace of t invariants, considered as a representation of $Z_G(t)$. Similarly $\mathcal{T}^t_{G,\mathbf{N}} \cong \mathcal{T}_{Z_G(t),\mathbf{N}^t}$.

Proof. For the Grassmann part, it is known that $\operatorname{Gr}^t_G \cong \operatorname{Gr}_{Z_G(t)}$. (We are unable to find an exact reference, but it is implicit in [BFGM02, §6]: $Z_G(t)$ is a Levi subgroup L in a parabolic P with radical U. The Grassmannian Gr_G is a disjoint union of "semiinfinite orbits": the connected components of $\operatorname{Gr}_P = L(\mathcal{K}) \cdot U(\mathcal{K})/L(\mathcal{O}) \cdot U(\mathcal{O})$. Since t acts trivially on L and contracts U to the origin, its fixed points on Gr_P are $L(\mathcal{K})/L(\mathcal{O}) = \operatorname{Gr}_L$.)

For the representation part, it is clear from the embedding $\mathcal{R}_{G,\mathbf{N}} \subset \operatorname{Gr}_G \times \mathbf{N}_\mathcal{O}$. $\square$

From this description, it is natural to introduce the following definition, which is given directly in terms of $(G, \mathbf{N})$ without reference to $\mathcal{R}_{G,\mathbf{N}}$.

Definition 5.2. Fix a maximal torus T of G as before. A *generalized root* α for a pair $(G, \mathbf{N})$ is either (I) a nonzero weight of $\mathbf{N}$ or (II) a root of $\operatorname{Lie} G$.

Generalized roots define hyperplanes in $\mathfrak{t} \overset{\text{def.}}{=} \operatorname{Lie} T$. Let $\mathfrak{t}^\circ$ denote the complement of the union of all generalized root hyperplanes.

From the above lemma, the fixed point subset $\mathcal{R}^t_{G,\mathbf{N}}$ of t is strictly larger than the fixed point $\mathcal{R}^T_{G,\mathbf{N}}$ of T if and only if $\langle \alpha, t \rangle = 0$ for some generalized root α. Hence $\mathfrak{t}^\circ$ consists of t such that $\mathcal{R}^t_{G,\mathbf{N}} = \mathcal{R}^T_{G,\mathbf{N}}$.

We call a nonzero weight α of $\mathbf{N}$ as a generalized root of type (I). More precisely, we say α is of type (I) further if there is no roots of G in $\mathbb{Q}\alpha$. We may assume α is primitive, i.e., it is not a positive integer multiple of another integral weight. Suppose $\langle t, \alpha \rangle = 0$ and t is not contained in any other generalized root hyperplanes. Then $Z_G(t) = T$, $\mathbf{N}^t = \mathbf{N}^T \oplus \bigoplus_{m \in \mathbb{Z}} \mathbf{N}(m\alpha)$, where $\mathbf{N}(m\alpha)$ (resp. $\mathbf{N}^T$) is the weight $m\alpha$ (resp. 0) subspace. We understand that $\mathbf{N}(m\alpha) = 0$ if $m\alpha$ is not a weight of $\mathbf{N}$. The first factor $\mathbf{N}^T$ plays no role by §3(vii)3(vii). Since $Z_G(t)$ is abelian, we already know $H^{Z_G(t)}_*(\mathcal{R}_{Z_G(t),\mathbf{N}^t})$ thanks to §4.

Other generalized roots are of type (II). They are just roots of G. We further suppose t is not contained in any other generalized root hyperplanes.

In particular, $\langle \mu, t \rangle \neq 0$ for any nonzero weight $\mu \notin \mathbb{Q}\alpha$. It could happen that a multiple of α is also a weight of $\mathbf{N}$. Let $\mathbf{N}(m\alpha)$ be the weight $m\alpha$ subspace, understanding it is 0 if $m\alpha$ is not a weight. Then $Z_G(t)$ is of semisimple rank 1 and $\mathbf{N}^t = \mathbf{N}^T \oplus \bigoplus_{m \in \mathbb{Q}} \mathbf{N}(m\alpha)$. For example, if $(G, \mathbf{N}) = (\mathrm{GL}_r, \mathfrak{gl}_r)$ and $t = \mathrm{diag}(t_1, t_1, t_3, \ldots, t_r)$ with $t_1, t_3, \ldots, t_r$ distinct, we have $Z_G(t) = \mathrm{GL}_2 \times T^{r-2}$, $\mathbf{N}^t = \mathfrak{gl}_2 \oplus \mathbb{C}^{r-2}$. Another example is $(G, \mathbf{N}) = (\mathrm{GL}_r, \mathbb{C}^r)$ with the same t. We have $\mathbf{N}^t = \{0\}$.

5(ii). Torus equivariant homology

The $T_{\mathcal{O}}$-equivariant Borel-Moore homology group $H_*^{T_{\mathcal{O}}}(\mathcal{R})$ is defined in the same way as in §2(ii). The same applies also for $T_{\mathcal{O}} \rtimes \mathbb{C}^\times$-equivariant homology. Recall $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ (resp. $H_*^{T_{\mathcal{O}}}(\mathcal{R})$) is a module over $H_G^*(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}/W]$ (resp. $H_T^*(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}]$). Also W acts on $H_*^{T_{\mathcal{O}}}(\mathcal{R})$ induced by the $N(T)$-action on $\mathcal{R}$, where $N(T)$ is the normalizer of T in G as usual (and $W = N(T)/T$).

Lemma 5.3. *The $H_T^*(\mathrm{pt})$-module $H_*^{T_{\mathcal{O}}}(\mathcal{R})$ is flat, and the $H_G^*(\mathrm{pt})$-module $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is flat. Moreover, the natural map $H_T^*(\mathrm{pt}) \otimes_{H_G^*(\mathrm{pt})} H_*^{G_{\mathcal{O}}}(\mathcal{R}) \to H_*^{T_{\mathcal{O}}}(\mathcal{R})$ is an isomorphism, and $H_*^{G_{\mathcal{O}}}(\mathcal{R}) = (H_*^{T_{\mathcal{O}}}(\mathcal{R}))^W$. The same applies for $T_{\mathcal{O}} \rtimes \mathbb{C}^\times$ and $G_{\mathcal{O}} \rtimes \mathbb{C}^\times$ equivariant homology groups.*

Proof. Same as the one of [BFM05, Lemma 6.2]   □

5(iii). Bimodule

Let us consider the diagram (3.2), and we restrict the $G_{\mathcal{O}} \times G_{\mathcal{O}}$ (resp. $G_{\mathcal{O}}$) action on the first and second (resp. third and fourth) columns to $T_{\mathcal{O}} \times G_{\mathcal{O}}$ (resp. $T_{\mathcal{O}}$). Then we have a right $H_*^{G_{\mathcal{O}}}(\mathcal{R})$-module structure on $H_*^{T_{\mathcal{O}}}(\mathcal{R})$, i.e., we have $c_1 * (c_2 * c_3) = (c_1 * c_2) * c_3$ for $c_1 \in H_*^{T_{\mathcal{O}}}(\mathcal{R})$, $c_2, c_3 \in H_*^{G_{\mathcal{O}}}(\mathcal{R})$. This is obvious from the proof of the associativity in Theorem 3.10, where we restrict $G_{\mathcal{O}}$-action to $T_{\mathcal{O}}$ at appropriate places.

Let $\mathbf{N}_T$ be the restriction of the G-module $\mathbf{N}$ to $T \subset G$. Then we can introduce the space $\mathcal{R}_{T, \mathbf{N}_T}$ of triples for T. It is nothing but the preimage in $\mathcal{R} \equiv \mathcal{R}_{G, \mathbf{N}}$ of $\mathrm{Gr}_T \subset \mathrm{Gr}_G$ under the natural projection $\mathcal{R} \to \mathrm{Gr}_G$. We modify the diagram (3.2) to
(5.4)
$$\mathcal{T}_{T, \mathbf{N}_T} \times \mathcal{R}_{G, \mathbf{N}} \xleftarrow{\ p\ } T_{\mathcal{K}} \times \mathcal{R}_{G, \mathbf{N}} \xrightarrow{\ q\ } T_{\mathcal{K}} \times_{T_{\mathcal{O}}} \mathcal{R}_{G, \mathbf{N}} \xrightarrow{\ m\ } \mathcal{T}_{G, \mathbf{N}}.$$

(We only write the bottom row, as the top row is given by closed embeddings.) We have $T_{\mathcal{O}} \times T_{\mathcal{O}}$ (resp. $T_{\mathcal{O}}$) action on the first and second

(resp. third and fourth) spaces. Then the diagram gives a left $H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})$-module structure on $H_*^{T_{\mathcal{O}}}(\mathcal{R})$, i.e., we have $c_1 * (c_2 * c_3) = (c_1 * c_2) * c_3$ for $c_1, c_2 \in H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})$, $c_3 \in H_*^{T_{\mathcal{O}}}(\mathcal{R})$. This again follows from the proof of the associativity in Theorem 3.10 with appropriate changes of spaces.

Two actions are commuting, i.e., $c_1 * (c_2 * c_3) = (c_1 * c_2) * c_3$ for $c_1 \in H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})$, $c_2 \in H_*^{T_{\mathcal{O}}}(\mathcal{R})$, $c_3 \in H_*^{G_{\mathcal{O}}}(\mathcal{R})$. This also follows from the proof of the associativity in Theorem 3.10 with appropriate changes of spaces. Therefore

Lemma 5.5. *The convolution product*

$$H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T}) \otimes H_*^{T_{\mathcal{O}}}(\mathcal{R}) \otimes H_*^{G_{\mathcal{O}}}(\mathcal{R}) \to H_*^{T_{\mathcal{O}}}(\mathcal{R})$$

gives an $(H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T}), H_*^{G_{\mathcal{O}}}(\mathcal{R}))$-bimodule structure on $H_*^{T_{\mathcal{O}}}(\mathcal{R})$. The same applies for $T_{\mathcal{O}} \rtimes \mathbb{C}^\times$ and $G_{\mathcal{O}} \rtimes \mathbb{C}^\times$ equivariant homology groups.

Let us remark that the convolution product $c * c'$ may not be linear in the second variable c', and is *not* so if we include the rotation $\mathbb{C}^\times$-action. See the computation in §3(vii)3(vii).

Also the W-action on $H_*^{T_{\mathcal{O}}}(\mathcal{R})$ commutes with the right action of $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ and normalizes the left action of $H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})$. Indeed, we have

Lemma 5.6. *We have a W-action on $H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})$ so that its algebra structure and its left module structure on $H_*^{T_{\mathcal{O}}}(\mathcal{R})$ in Lemma 5.5 are W-equivariant. The same applies to $T_{\mathcal{O}} \rtimes \mathbb{C}^\times$ equivariant homology groups.*

Proof. Let $N(T)$ be the normalizer of T. The $N(T)$-action on $\mathcal{R}$ preserves $\mathcal{R}_{T,\mathbf{N}_T}$. Hence we have the induced W-action on $H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})$. Diagrams (3.2) for $(T, \mathbf{N}_T)$ and (5.4), used to define convolution products, are $N(T)$-equivariant. Therefore the convolution products are W-equivariant. $\qquad\square$

Let us consider e, the fundamental class of of the fiber of $\mathcal{R} \to \mathrm{Gr}_G$ at the base point $[1] \in \mathrm{Gr}_G$ as in Theorem 3.10. It was the unit in $H_*^{G_{\mathcal{O}}}(\mathcal{R})$, but we consider it as an element in the bimodule $H_*^{T_{\mathcal{O}}}(\mathcal{R})$ instead.

Lemma 5.7. (1) *The left multiplication $H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T}) \ni c \mapsto c * e \in H_*^{T_{\mathcal{O}}}(\mathcal{R})$ is the pushforward homomorphism ι_* for the embedding $\iota\colon \mathcal{R}_{T,\mathbf{N}_T} \to \mathcal{R}$.*

(2) *The right multiplication $H_*^{G_{\mathcal{O}}}(\mathcal{R}) \ni c' \mapsto e * c' \in H_*^{T_{\mathcal{O}}}(\mathcal{R})$ is the homomorphism given by the restriction from $G_{\mathcal{O}}$ to $T_{\mathcal{O}}$. In particular, it is $H_G^*(\mathrm{pt})$-linear under the forgetting homomorphism $H_G^*(\mathrm{pt}) \to H_T^*(\mathrm{pt})$.*

(3) *Both (1), (2) are true for $T_{\mathcal{O}} \rtimes \mathbb{C}^\times$, $G_{\mathcal{O}} \rtimes \mathbb{C}^\times$ equivariant homology groups.*

The proof is exactly the same as the proof that e is unit in Theorem 3.10. By (2), we have a well-defined homomorphism

(5.8) $$e * \bullet \colon H_*^{G_\mathcal{O}}(\mathcal{R}) \otimes_{H_G^*(\mathrm{pt})} H_T^*(\mathrm{pt}) \xrightarrow{\cong} H_*^{T_\mathcal{O}}(\mathcal{R}).$$

It is an isomorphism thanks to Lemma 5.3.

Lemma 5.9. *The homomorphism ι_* becomes an isomorphism over $\mathfrak{t}^\circ$, the complement of the union of all generalized root hyperplanes. In particular, it is injective.*

For $T_\mathcal{O} \rtimes \mathbb{C}^\times$-equivariant homology groups, ι_* is an isomorphism over $\mathfrak{t}^\circ \times \mathrm{Lie}(\mathbb{C}^\times)$.

Proof. The last assertion is a consequence of the first and Lemma 5.3.

By Lemma 5.1 (or a direct consideration), we have $\mathcal{R}_{T,\mathbf{N}_T}^T = \mathcal{R}^T$. The localization theorem for equivariant homology groups implies the assertion. $\square$

Hereafter we often use the notation '$|_{\mathfrak{t}^\circ}$' meaning the localization at the ideal given by generalized root hyperplanes, i.e., tensor product with $\mathbb{C}[\mathfrak{t}^\circ]$ over $H_T^*(\mathrm{pt})$.

Lemma 5.10. (1) $\iota_* \colon H_*^{T_\mathcal{O}}(\mathcal{R}_{T,\mathbf{N}_T})^W \to H_*^{G_\mathcal{O}}(\mathcal{R})$ *is an algebra homomorphism.*

(2) *The same is true for $T_\mathcal{O} \rtimes \mathbb{C}^\times$ and $G_\mathcal{O} \rtimes \mathbb{C}^\times$ equivariant homology groups. In particular, ι_* in (1) respects the Poisson structures.*[5]

Proof. Our idea of the proof is based on [CKL13, §5.3].

Let c_1, $c_2 \in H_*^{T_\mathcal{O}}(\mathcal{R}_{T,\mathbf{N}_T})$. By Lemma 5.7(1) $\iota_*(c_a) = c_a * e$ $(a = 1, 2)$. If c_a is W-invariant, $c_a * e$ is also by Lemma 5.6, hence we have $c_a * e = e * c_a'$ for some $c_a' \in H_*^{G_\mathcal{O}}(\mathcal{R})$ by Lemma 5.7(2). The map ι_* in the statement is nothing but $c_a \mapsto c_a'$.

Now the associativity implies

$$(c_1 * c_2) * e = c_1 * (c_2 * e) = c_1 * (e * c_2')$$
$$= (c_1 * e) * c_2' = (e * c_1') * c_2' = e * (c_1' * c_2').$$

It means that ι_* is an algebra homomorphism. The argument works also when the loop rotation $\mathbb{C}^\times$ is included. $\square$

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[5]The statement will make sense after we prove the commutativity in Proposition 5.15 below. The same applies to Lemma 5.11(3).

5(iv). From the variety of triples to the affine Grassmannian

In the same (and simpler) way as in §5(iii), we have a natural $(H_*^{T_\mathcal{O}}(\mathrm{Gr}_T),$ $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G))$-module structure on $H_*^{T_\mathcal{O}}(\mathrm{Gr}_G)$. Alternatively it is a special case of the above construction with $\mathbf{N} = 0$.

Let $\mathbf{z}\colon \mathrm{Gr}_G \to \mathcal{T}$ be the closed embedding given by considering Gr_G as the 0 section of the vector bundle $\mathcal{T}$. We have $\mathbf{z}^{-1}(\mathcal{R}) = \mathrm{Gr}_G$. Since $\mathcal{T} \to \mathrm{Gr}_G$ is a vector bundle, we have the pull-back homomorphism $\mathbf{z}^* \omega_\mathcal{T} \to \omega_{\mathrm{Gr}_G}[2\dim \mathbf{N}_\mathcal{O}]$, where $\dim \mathbf{N}_\mathcal{O}$ is the rank of the vector bundle. Let $\tilde{\mathbf{z}}$ denote the inclusion $\mathrm{Gr}_G \to \mathcal{R}$ (so that $\mathbf{z} = i \circ \tilde{\mathbf{z}}$). We have the pull-back with support homomorphisms $\mathbf{z}^*\colon \omega_\mathcal{R}[-2\dim \mathbf{N}_\mathcal{O}] \to \tilde{\mathbf{z}}_* \omega_{\mathrm{Gr}_G}$ and $\mathbf{z}^*\colon H_*^{G_\mathcal{O}}(\mathcal{R}) \to H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$, where the degree is given relative to $2\dim \mathbf{N}_\mathcal{O}$ for the former so that the degree is preserved. In fact, it is also easy to check that $\mathbf{z}^*$ is the same as the composite of $H_*^{G_\mathcal{O}}(\mathcal{R}) \xrightarrow{i_*} H_*^{G_\mathcal{O}}(\mathcal{T}) \xrightarrow[\cong]{\mathbf{z}^*} H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$. The second $\mathbf{z}^*$ is an isomorphism and its inverse is π^* where $\pi\colon \mathcal{T} \to \mathrm{Gr}_G$ is the projection.

We also have $\mathbf{z}^*\colon H_*^{T_\mathcal{O}}(\mathcal{R}) \to H_*^{T_\mathcal{O}}(\mathrm{Gr}_G)$, and $\mathbf{z}^*\colon H_*^{T_\mathcal{O}}(\mathcal{R}_{T,\mathbf{N}_T}) \to H_*^{T_\mathcal{O}}(\mathrm{Gr}_T)$. The second $\mathbf{z}$ should be understood as $\mathrm{Gr}_T \to \mathcal{T}_{T,\mathbf{N}_T}$, but is denoted by the same letter for brevity.

Lemma 5.11. (1) $\mathbf{z}^*\colon H_*^{G_\mathcal{O}}(\mathcal{R}) \to H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ *is an algebra homomorphism.* *The same is true for* $\mathbf{z}^*\colon H_*^{T_\mathcal{O}}(\mathcal{R}_{T,\mathbf{N}_T}) \to H_*^{T_\mathcal{O}}(\mathrm{Gr}_T)$.

(2) $\mathbf{z}^*\colon H_*^{T_\mathcal{O}}(\mathcal{R}) \to H_*^{T_\mathcal{O}}(\mathrm{Gr}_G)$ *is a homomorphism of bimodules.*

(3) *Both* (1), (2) *are true for* $T_\mathcal{O} \rtimes \mathbb{C}^\times$, $G_\mathcal{O} \rtimes \mathbb{C}^\times$ *equivariant homology groups. In particular,* $\mathbf{z}^*$ *in* (1) *respects the Poisson structures.*

Proof. We give the proof of the first statement of (1). The second statement is the special case $G = T$. The proof of (2) is straightforward modification of the proof of (1), and is omitted. The proof of (3) is the same as (1), just check that everything is $\mathbb{C}^\times$-equivariant.

The idea of the proof is similar to one of Lemma 5.10.

We consider the diagram

$$(5.12) \qquad \mathcal{T} \times \mathrm{Gr}_G \longleftarrow G_\mathcal{K} \times \mathrm{Gr}_G \longrightarrow G_\mathcal{K} \times_{G_\mathcal{O}} \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_G,$$

which is nothing but the diagram (3.1) with the leftmost term replaced by $\mathcal{T} \times \mathrm{Gr}_G$ via the inclusion $\mathrm{Gr}_G \times \mathrm{Gr}_G \xrightarrow{\mathbf{z} \times \mathrm{id}_{\mathrm{Gr}_G}} \mathcal{T} \times \mathrm{Gr}_G$. As in §5(iii) we can view $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ as a left $H_*^{G_\mathcal{O}}(\mathcal{R})$-module. Then we have $c \bar{*} c' = \mathbf{z}^*(c) * c'$, where the left $\bar{*}$ is the module action, and the right $*$ is the multiplication in $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$. Now we take $c' = 1$, the unit of $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$. Then the

associativity implies

$$\mathbf{z}^*(c_1 * c_2) = (c_1 * c_2)\bar{*}1 = c_1\bar{*}(c_2\bar{*}1) = c_1\bar{*}\mathbf{z}^*(c_2) = \mathbf{z}^*(c_1) * \mathbf{z}^*(c_2).$$

This means that $\mathbf{z}^*$ is an algebra homomorphism. The proof of the associativity is the same as one in Theorem 3.10, hence is omitted. □

Lemma 5.13. *The homomorphisms $\mathbf{z}^*$ in three cases in Lemma 5.11 become isomorphisms over $\mathfrak{t}^\circ$, the complement of the union of all generalized root hyperplanes. In particular, they are injective.*

Proof. Since $H_*^{G_\mathcal{O}}(\bullet) = (H_*^{T_\mathcal{O}}(\bullet))^W$ for both $\bullet = \mathcal{R}$ and Gr_G, it is enough to check the assertion for $\mathbf{z}^*$ in (2). (The second $\mathbf{z}^*$ in (1) is the special case $G = T$.)

Recall that $\mathbf{z}^*$ is the composite of $H_*^{T_\mathcal{O}}(\mathcal{R}) \xrightarrow{i_*} H_*^{T_\mathcal{O}}(\mathcal{T}) \xrightarrow[\cong]{\mathbf{z}^*} H_*^{T_\mathcal{O}}(\mathrm{Gr}_G)$, as we remarked above. Therefore it is enough to show that i_* becomes an isomorphism over $\mathfrak{t}^\circ$. The pushforward homomorphisms of inclusions $\mathcal{R}^T \subset \mathcal{R}$, $\mathcal{T}^T \subset \mathcal{T}$ induce isomorphisms $H_*^{T_\mathcal{O}}(\mathcal{R}^T)|_{\mathfrak{t}^\circ} \xrightarrow{\cong} H_*^{T_\mathcal{O}}(\mathcal{R})|_{\mathfrak{t}^\circ}$, $H_*^{T_\mathcal{O}}(\mathcal{T}^T)|_{\mathfrak{t}^\circ} \xrightarrow{\cong} H_*^{T_\mathcal{O}}(\mathcal{T})|_{\mathfrak{t}^\circ}$ respectively. Therefore it is enough to show that i_* for $H_*^{T_\mathcal{O}}(\mathcal{R}^T)|_{\mathfrak{t}^\circ} \to H_*^{T_\mathcal{O}}(\mathcal{T}^T)|_{\mathfrak{t}^\circ}$ is an isomorphism. But this is clear from Lemma 5.1 : $\mathcal{R}^T \cong \mathcal{R}_{T,\mathbf{N}^T} \cong \mathrm{Gr}_G \times \mathbf{N}_\mathcal{O}^T \cong \mathcal{T}^T$. □

Remark 5.14. Let $\mathbf{V}$ be another representation. Then there is a natural embedding $H_*^{G_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}\oplus\mathbf{V}}) \hookrightarrow H_*^{G_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}})$ given by the pull-back homomorphism with respect to the embedding $\mathcal{T}_{T,\mathbf{N}} \subset \mathcal{T}_{T,\mathbf{N}\oplus\mathbf{V}}$ of a subbundle. (cf. §4(vi)) This can be proved by the same argument, or by observing $H_*^{G_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}\oplus\mathbf{V}}) \to H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ factors as $H_*^{G_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}\oplus\mathbf{V}}) \to H_*^{G_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}}) \to H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$.

5(v). Classical description of Coulomb branches

We study $\mathcal{M}_C$ over $\mathfrak{t}^\circ$ in this subsection. This is called a classical description of the Coulomb branch $\mathcal{M}_C$ in the physics literature. This description first leads to a proof of the commutativity of $H_*^{G_\mathcal{O}}(\mathcal{R})$, as we promised in §3(iv).

Proposition 5.15. $H_*^{G_\mathcal{O}}(\mathcal{R})$ *is commutative.*

One way to prove this is to use Lemma 5.13 and the commutativity of $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$, proved in [BFM05]. The proof of the commutativity of $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$, as was explained in §3(iv), uses the Beilinson-Drinfeld Grassmannian. This proof will be given at a categorical level in [Affine, §3]. We

present another argument, completely avoiding Beilinson-Drinfeld Grassmannian now.

Proof of Proposition 5.15. Since we have proved that $H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})$ is commutative (see §4), the multiplication is $H_{T_{\mathcal{O}}}^*(\mathrm{pt})$-linear in both first and second variables. In particular, its localization $H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})|_{\mathfrak{t}^{\circ}}$ inherits the algebra structure.

Recall the embedding $\iota \colon \mathcal{R}_{T,\mathbf{N}_T} \to \mathcal{R}$. By Lemma 5.9 it induces an isomorphism between the localized equivariant homology groups:

$$\iota_* \colon H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})|_{\mathfrak{t}^{\circ}} \xrightarrow{\cong} H_*^{T_{\mathcal{O}}}(\mathcal{R})|_{\mathfrak{t}^{\circ}}.$$

We have a chain of injective maps

$$H_*^{G_{\mathcal{O}}}(\mathcal{R}) \to H_*^{T_{\mathcal{O}}}(\mathcal{R}) \to H_*^{T_{\mathcal{O}}}(\mathcal{R})|_{\mathfrak{t}^{\circ}} \xrightarrow[\sim]{\iota_*^{-1} = (\bullet * e)^{-1}} H_*^{T_{\mathcal{O}}}(\mathcal{R}_{T,\mathbf{N}_T})|_{\mathfrak{t}^{\circ}},$$

where the injectivity of the first two maps follows from Lemma 5.3. By the proof of Lemma 5.10, the composite respects the multiplication. Therefore $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is commutative. $\qquad\qquad\square$

We endow $H_*^{T_{\mathcal{O}}}(\mathcal{R}) \cong H_*^{G_{\mathcal{O}}}(\mathcal{R}) \otimes_{H_G^*(\mathrm{pt})} H_T^*(\mathrm{pt})$ with an algebra structure induced from $H_*^{G_{\mathcal{O}}}(\mathcal{R})$. Note that *a priori* it does not have an algebra structure. The multiplication $*$ on $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is $H_G^*(\mathrm{pt})$-linear only in the first variable. In our case, we have just proved that $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is commutative, and hence also linear in the second variable. Therefore the multiplication is well-defined. This construction does not work in general, say $H_*^{T_{\mathcal{O}} \rtimes \mathbb{C}^{\times}}(\mathcal{R})$, as it is noncommutative.

Remark 5.16. Let us emphasize further how the commutativity of $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is important in this construction.

Consider the usual finite dimensional Steinberg variety St, and its analog $\overline{\mathrm{St}}$. We also consider $\mathrm{St}_P, \overline{\mathrm{St}}_P$ corresponding to a parabolic subgroup P. Our $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ is an analog of $H_*^P(\overline{\mathrm{St}}_P)$, which is isomorphic to $H_*^G(\mathrm{St}_P)$. In this situation, $H_*^P(\overline{\mathrm{St}}_P)$ is an algebra, but $H_*^T(\overline{\mathrm{St}}_P)$ is not in general. It is clear from the following: We have $H_*^P(\overline{\mathrm{St}}_P) \cong e_P H_*^B(\overline{\mathrm{St}}) e_P$ for an idempotent e_P. Then $H_*^T(\overline{\mathrm{St}}_P) \cong H_*^B(\overline{\mathrm{St}}) e_P$. Thus $H_*^T(\overline{\mathrm{St}}_P)$ is a bimodule, but not an algebra.

We have

Lemma 5.17. *The pushforward homomorphism ι_* of the embedding $\iota:$ $\mathcal{R}_{T,\mathbf{N}_T} \to \mathcal{R}$, composed with the inverse of* (5.8), *gives an algebra homomorphism $H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}_T}) \to H_*^{T\circ}(\mathcal{R}) \cong H_*^{G\circ}(\mathcal{R}) \otimes_{H_G^*(\mathrm{pt})} H_T^*(\mathrm{pt})$. It becomes an isomorphism over $\mathfrak{t}^\circ$. In particular, it is injective.*

Thus $H_*^{G\circ}(\mathcal{R})$ and $H_*^{T\circ}(\mathrm{Gr}_T)$ are related as

$$(5.18) \qquad H_*^{T\circ}(\mathrm{Gr}_T)^W \xleftarrow{\mathbf{z}^*} H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}_T})^W \xrightarrow{\iota_*} H_*^{T\circ}(\mathcal{R})^W \cong H_*^{G\circ}(\mathcal{R}).$$

Both ι_* and $\mathbf{z}^*$ are algebra homomorphisms, and become isomorphisms over $\mathfrak{t}^\circ$. (See Lemma 5.13 for the last assertion.)

Proposition 5.19. (1) *We have a $\mathbb{C}[\mathfrak{t}]^W$-algebra isomorphism*

$$H_*^{T\circ}(\mathrm{Gr}_T)^W \cong \mathbb{C}[\mathfrak{t} \times T^\vee]^W,$$

where $T^\vee$ is the dual torus of T. The W-action on $\mathfrak{t} \times T^\vee$ is the usual one.
  (2) *The quantized algebra $H_*^{T\circ \rtimes \mathbb{C}^\times}(\mathrm{Gr}_T)^W$ is isomorphic to the W-invariant part of the ring of $\hbar$-differential operators $D_\hbar(T^\vee)$ on $T^\vee$. The homomorphism $H_{G_\mathcal{O} \rtimes \mathbb{C}^\times}^*(\mathrm{pt}) \to H_*^{T\circ \rtimes \mathbb{C}^\times}(\mathrm{Gr}_T)^W$ is given by invariant vector fields via the isomorphism $H_{G_\mathcal{O} \rtimes \mathbb{C}^\times}^*(\mathrm{pt}) \cong H_{T \times \mathbb{C}^\times}^*(\mathrm{pt})^W \cong \mathrm{Sym}(\mathfrak{t}^\vee)^W[\hbar]$.*

Proof. Since Gr_T consists of points parametrized by the coweight lattice Y of T, we have $H_*^{T\circ}(\mathrm{Gr}_T) \cong \mathbb{C}[\mathfrak{t} \times T^\vee]$. (This is a special case of Theorem 4.1.) The W-action on $H_*^{T\circ}(\mathrm{Gr}_T)$ is given by the $N(T)$-action on Gr_T, and the induced $W = N(T)/T$-action on $\mathrm{Gr}_T \times_T EG \cong Y \times (EG/T) = Y \times BT$, where $EG \to BG$, $ET \to BT$ are the classifying spaces for G and T. This induces the usual W-action on $\mathfrak{t} \times T^\vee$.

The argument for the quantized version is the same. $\square$

Taking spectrum of (5.18), Lemma 5.11, we get

$$(5.20) \qquad \begin{aligned} &\mathfrak{t} \times T^\vee/W \to \mathcal{M}_C(T, \mathbf{N}_T)/W \leftarrow \mathcal{M}_C(G, \mathbf{N}), \\ &\mathcal{M}_C(G, \mathbf{N}) \leftarrow \mathcal{M}_C(G, 0). \end{aligned}$$

Since homomorphisms are injective, those morphisms are *dominant*.

When $\mathbf{N} = 0$, the left and right morphisms are the identity. The middle morphism is an affine blowup described in [BFM05, §2.5]. This is not explicitly stated in [BFM05], but is clear from the proof of [BFM05, Th. 2.12].

Next consider examples in §4(iv) (hence $G = T$). The middle morphism is the identity. The left (and the right) morphism is given by an open embedding $\mathbb{C} \times \mathbb{C}^\times \ni (a, x) \mapsto (a, x, y = a^N x^{-1}) \in \{xy = a^N\}$.

Corollary 5.21. (1) *We have a birational isomorphism*

$$\mathcal{M}_C(G, \mathbf{N}) \approx \mathfrak{t} \times T^\vee / W$$

given by $\mathbf{z}^* \iota_*^{-1}$, *where* ϖ *in* (3.17) *corresponds to the first projection in* $\mathfrak{t} \times T^\vee / W$. *It is an isomorphism over* $\mathfrak{t}^\circ \times T^\vee / W$. *In particular, the generic fiber of* ϖ *is* $T^\vee$.[6]

(2) *Moreover the Poisson structure on* $\mathcal{M}_C(G, \mathbf{N})|_{\varpi^{-1}(\mathfrak{t}^\circ / W)}$ *corresponds to the standard one on* $\mathfrak{t}^\circ \times T^\vee / W$, *the restriction of the symplectic structure on* $T^* T^\vee / W$ *to the open subset* $\mathfrak{t}^\circ \times T^\vee / W$.

Corollary 5.22. $\mathcal{M}_C(G, \mathbf{N})$ *is an integral scheme.*

Proof. We have an injective homomorphism $H_*^{G_\mathcal{O}}(\mathcal{R}) \to \mathbb{C}[\mathfrak{t}^\circ \times T^\vee]$. The latter is an integral domain. Therefore $H_*^{G_\mathcal{O}}(\mathcal{R})$ is also. $\square$

Remark 5.23. Let us consider the open subvariety

$$\varpi^{-1}(\mathfrak{t} \setminus \{\text{root hyperplanes}\}/W)$$

in $\mathcal{M}_C$. The morphism $\mathcal{M}_C(G, \mathbf{N}) \to \mathcal{M}_C(T, \mathbf{N}_T)/W$ in (5.20) becomes an isomorphism over this open subvariety, thanks to the localization theorem in equivariant homology groups. Therefore $\mathbb{C}[\varpi^{-1}(\mathfrak{t} \setminus \{\text{root hyperplanes}\}/W)]$ has an explicit presentation from that of $\mathcal{A}(T, \mathbf{N}_T) = \mathbb{C}[\mathcal{M}_C(T, \mathbf{N}_T)]$. Furthermore, we have an embedding $\mathcal{A}(T, \mathbf{N}_T) \hookrightarrow \mathcal{A}(T, 0)$, where the latter is just a Laurent polynomial ring $\mathbb{C}[\mathfrak{t} \times T^\vee]$.

These embeddings make sense for quantized Coulomb branches: we have $\mathcal{A}_\hbar(G, \mathbf{N}) \hookrightarrow \mathcal{A}_\hbar(T, \mathbf{N}_T)[\hbar^{-1}, (\text{root} + m\hbar)^{-1}]_{m \in \mathbb{Z}} \hookrightarrow \mathcal{A}_\hbar(T, 0)[\hbar^{-1}, (\text{root} + m\hbar)^{-1}]_{m \in \mathbb{Z}}$. Here one can easily check that the multiplicative subset generated by $\hbar, \text{roots} + \mathbb{Z}\hbar$ satisfies the Ore condition in $\mathcal{A}_\hbar(T, \mathbf{N}_T)$ and $\mathcal{A}_\hbar(T, 0)$ thanks to Lemma 3.20. Therefore the localization as $H_T^*(\text{pt})$-modules have algebra structure. Moreover $\mathcal{A}_\hbar(T, \mathbf{N}_T)[\hbar^{-1}, (\text{root} + m\hbar)^{-1}]_{m \in \mathbb{Z}}$ is the (localized) ring of $\hbar$-difference operators on $\mathfrak{t}$.

The ring homomorphism $\iota_* \colon \mathcal{A}_\hbar(T, \mathbf{N}_T)^W \to \mathcal{A}_\hbar$ becomes an isomorphism if we invert the expressions $\hbar, \alpha + m\hbar$ where α is a root of G and m is an integer, considered as elements in $H_{T \times \mathbb{C}^\times}^*(\text{pt})$ (5(iii)). Indeed, it suffices to check that for any dominant coweight λ, the homomorphism of $(H_{T \times \mathbb{C}^\times}^*(\text{pt}))^W$-modules $\iota_* \colon (H_*^{T_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}_{T, \mathbf{N}_T})_{W\lambda})^W \to H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}_\lambda)$ becomes an isomorphism after inverting the expressions $\hbar, \alpha + m\hbar$. To this end

---

[6]The third named author thanks Kentaro Hori for correcting his mistake.

note that at a T-fixed point $w\lambda \in \mathcal{R}_\lambda$, $w \in W$, the quotient of the tangent space to $\mathcal{R}_\lambda$ modulo the tangent space to $\mathcal{R}_{T,\mathbf{N}_T}$ equals the tangent space to Gr_G^λ, and the weights of $T \times \mathbb{C}^\times$ in the latter tangent space are all of the form $\hbar, \alpha + m\hbar$.

We further have a ring homomorphism $\mathbf{z}^* \colon \mathcal{A}_\hbar(T, \mathbf{N}_T) \to \mathcal{A}_\hbar(T, 0)$ (5(iv)). By 5.19 $\mathcal{A}_\hbar(T, 0)$ is a $\mathbb{C}[\hbar]$-algebra generated by w_r, $\mathsf{u}_r^{\pm 1}$ ($1 \leq r \leq \dim T$) with relations $[\mathsf{u}_r^{\pm 1}, w_s] = \pm\delta_{r,s}\hbar\mathsf{u}_r^{\pm 1}$. (Here we take coordinates of $T^\vee$ and the induced coordinates on $\mathfrak{t}$.)

In [BDG15, §4] it is argued that $\mathbb{C}[\mathcal{M}_C]$ is embedded into an explicit combinatorial ring, denoted by $\mathbb{C}[\mathcal{M}_C^{\mathrm{abel}}]$ from a physical intuition. It turns out that $\mathbb{C}[\mathcal{M}_C^{\mathrm{abel}}]$ is nothing but the coordinate ring of $\varpi^{-1}(\mathfrak{t} \setminus \{\text{root hyperplanes}\}/W)$. This is obvious from [BDG15, (4.11)]. The explicit presentation in [BDG15, (4.9)] coincides with one induced from $\mathcal{A}(T, \mathbf{N}_T)$ explained above. The quantized case is mentioned in [BDG15, §4.5].

5(vi). Flatness guarantees that codimension 1 is enough

Let $\mathfrak{t}^\bullet$ be the complement to all pairwise intersections of generalized root hyperplanes. We have $\mathrm{codim}_{\mathfrak{t}}(\mathfrak{t} \setminus \mathfrak{t}^\bullet) = 2$. We set $\mathfrak{t}^\bullet = \mathfrak{t}$ if $\dim \mathfrak{t} = 1$, and $\mathfrak{t}^\bullet = \mathfrak{t} \setminus \{0\}$ if $\dim \mathfrak{t} = 2$.

Let $t \in \mathfrak{t}^\bullet \setminus \mathfrak{t}^\circ$. Let $G' = Z_G(t)$ and $\mathbf{N}' = \mathbf{N}^t$, where $Z_G(t)$, $\mathbf{N}^t$ are as in Lemma 5.1. We consider T as a maximal torus of G'.

We consider the diagrams (5.18) for both $(G, \mathbf{N})$, $(G', \mathbf{N}')$ and will study their compatibilities. Let us denote the maps for $(G', \mathbf{N}')$ by $\mathbf{z}'^*$, ι'_*.

Note that the embedding $\mathbf{z} \colon \mathrm{Gr}_T \to \mathcal{T}_{T,\mathbf{N}_T}$ factors as $\mathrm{Gr}_T \xrightarrow{\mathbf{z}'} \mathcal{T}_{T,\mathbf{N}'_T} \xrightarrow{\mathbf{z}''} \mathcal{T}_{T,\mathbf{N}_T}$ where $\mathbf{z}''$ is an embedding of a subbundle. We have also another embedding $\mathbf{z}''' \colon \mathcal{T}_{G',\mathbf{N}'} \to \mathcal{T}_{G',\mathbf{N}}$ of a subbundle. We have pull-back with support homomorphisms $\mathbf{z}''^* \colon H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}_T}) \to H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}'_T})$ together with $\mathbf{z}^*$, $\mathbf{z}'^*$, $\mathbf{z}'''^*$ as in §5(iv).

Note also that $\iota \colon \mathcal{R}_{T,\mathbf{N}_T} \to \mathcal{R}$ factors as $\mathcal{R}_{T,\mathbf{N}_T} \to \mathcal{R}_{G',\mathbf{N}} \to \mathcal{R}$. Let ι'' (resp. ι''') denote the left (resp. right) map.

Let us consider the following diagram:

$$(5.24)$$

$$
\begin{array}{ccccc}
\mathbb{C}[\mathfrak{t} \times T^\vee] = H_*^{T\circ}(\mathrm{Gr}_T) & \xleftarrow{\;\mathbf{z}^*\;} & H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}_T}) & \xrightarrow{\;\iota_*\;} & H_*^{T\circ}(\mathcal{R}) \\
\| & & \| & & \uparrow{\scriptstyle\iota'''_*} \\
\mathbb{C}[\mathfrak{t} \times T^\vee] = H_*^{T\circ}(\mathrm{Gr}_T) & \xleftarrow{\;\mathbf{z}^*\;} & H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}_T}) & \xrightarrow{\;\iota''_*\;} & H_*^{T\circ}(\mathcal{R}_{G',\mathbf{N}}) \\
\| & & \downarrow{\scriptstyle\mathbf{z}''^*} & & \downarrow{\scriptstyle\mathbf{z}'''^*} \\
\mathbb{C}[\mathfrak{t} \times T^\vee] = H_*^{T\circ}(\mathrm{Gr}_T) & \xleftarrow{\;\mathbf{z}'^*\;} & H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}'_T}) & \xrightarrow{\;\iota'_*\;} & H_*^{T\circ}(\mathcal{R}_{G',\mathbf{N}'}).
\end{array}
$$

Lemma 5.25. *All squares are commutative.*

Proof. The commutativity is obvious except for the right bottom square. The commutativity of the right bottom square follows from the base change, noticing that $\mathbf{z}''^*$, $\mathbf{z}'''^*$ are $\mathbf{z}''^!$, $\mathbf{z}'''^!$ up to appropriate shifts respectively, and

$$
\begin{array}{ccc}
\mathcal{T}_{T,\mathbf{N}_T} & \longrightarrow & \mathcal{T}_{G',\mathbf{N}} \\
\uparrow & & \uparrow \\
\mathcal{T}_{T,\mathbf{N}'_T} & \longrightarrow & \mathcal{T}_{G',\mathbf{N}'}
\end{array}
$$

is Cartesian. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Theorem 5.26. *Let $\mathcal{M} \xrightarrow{\Pi} \mathfrak{t}/W$ be an affine scheme over $\mathfrak{t}/W$, and let $\mathcal{M}^\bullet := \Pi^{-1}(\mathfrak{t}^\bullet/W)$. We assume that the natural morphism $\Pi_*\mathcal{O}_\mathcal{M} \to j_*\Pi_*\mathcal{O}_{\mathcal{M}^\bullet}$ is an isomorphism where $j\colon \mathfrak{t}^\bullet/W \hookrightarrow \mathfrak{t}/W$ is an open embedding. We further assume $\mathcal{M}$ is equipped with the following data.*

1) Assume we are given an isomorphism between the localizations

$$
\mathbb{C}[\mathcal{M}] \otimes_{\mathbb{C}[\mathfrak{t}/W]} \mathbb{C}[\mathfrak{t}^\circ/W] \xrightarrow{\Xi} \mathbb{C}[\mathfrak{t} \times T^\vee]^W \otimes_{\mathbb{C}[\mathfrak{t}/W]} \mathbb{C}[\mathfrak{t}^\circ/W]
$$
$$
\xrightarrow{\sim} H_*^{G^\circ}(\mathcal{R}_{G,\mathbf{N}}) \otimes_{\mathbb{C}[\mathfrak{t}/W]} \mathbb{C}[\mathfrak{t}^\circ/W],
$$

(the second isomorphism is $\iota_*(\mathbf{z}^*)^{-1}$ of (5.18) plus Proposition 5.19). The composition is denoted

$$
\Xi^\circ\colon \mathbb{C}[\mathcal{M}] \otimes_{\mathbb{C}[\mathfrak{t}/W]} \mathbb{C}[\mathfrak{t}^\circ/W] \xrightarrow{\sim} H_*^{G^\circ}(\mathcal{R}_{G,\mathbf{N}}) \otimes_{\mathbb{C}[\mathfrak{t}/W]} \mathbb{C}[\mathfrak{t}^\circ/W].
$$

2) Let $t \in \mathfrak{t}^\bullet \setminus \mathfrak{t}^\circ$. Let $G' = Z_G(t)$ and $\mathbf{N}' = \mathbf{N}^t$, where $Z_G(t)$, $\mathbf{N}^t$ are as in Lemma 5.1. We consider T as a maximal torus of G'. Assume we are given an isomorphism

$$
\Xi^t\colon (\mathbb{C}[\mathcal{M}] \otimes_{\mathbb{C}[\mathfrak{t}/W]} \mathbb{C}[\mathfrak{t}])_t \xrightarrow{\cong} (H_*^{G'^\circ}(\mathcal{R}_{G',\mathbf{N}'}) \otimes_{H_{G'}^*(\mathrm{pt})} \mathbb{C}[\mathfrak{t}])_t
$$

such that

$$
\mathbf{z}'^*(\iota'_*)^{-1}\left(\Xi^t \otimes_{\mathbb{C}[\mathfrak{t}]_t} \mathbb{C}[\mathfrak{t}^\circ]\right) = \Xi \otimes_{\mathbb{C}[\mathfrak{t}^\circ/W]} \mathbb{C}[\mathfrak{t}^\circ].
$$

Here Ξ is a $\mathbb{C}[\mathfrak{t}^\circ/W]$-algebra isomorphism, while Ξ^t is a $\mathbb{C}[\mathfrak{t}]_t$-algebra isomorphism.

   Then Ξ° extends to an isomorphism between $\mathcal{M}$ and $\mathrm{Spec}(H_*^{G^\circ}(\mathcal{R}))$ as schemes over $\mathfrak{t}/W$.

This proposition enables us to determine $\mathcal{M}$ in two steps. We first determine the Coulomb branch $H_*^{G'_{\mathcal{O}}}(\mathcal{R}_{G',\mathbf{N}'})$ for another pair $(G',\mathbf{N}')$. Since G' has semisimple rank 1, it should be easier than $H_*^{G_{\mathcal{O}}}(\mathcal{R})$. Then we look for $\mathcal{M}$ so that various $H_*^{G'_{\mathcal{O}}}(\mathcal{R}_{G',\mathbf{N}'})$ are 'glued' to form a flat family.

Proof. Recall that for a dominant coweight λ we denote by $\mathcal{R}_\lambda$ (resp. $\mathcal{R}_{\leq\lambda}$) the preimage of the $G_{\mathcal{O}}$-orbit Gr_G^λ (resp. of its closure $\overline{\mathrm{Gr}}_G^\lambda$); we also have $\mathcal{R}_{<\lambda} := \mathcal{R}_{\leq\lambda} \setminus \mathcal{R}_\lambda$. Then the closed embedding $\mathcal{R}_{<\lambda} \hookrightarrow \mathcal{R}_{\leq\lambda}$ gives rise to the exact sequence $0 \to H_*^{G_{\mathcal{O}}}(\mathcal{R}_{<\lambda}) \to H_*^{G_{\mathcal{O}}}(\mathcal{R}_{\leq\lambda}) \to H_*^{G_{\mathcal{O}}}(\mathcal{R}_\lambda) \to 0$ (Lemma 2.6). Since $H_*^{G_{\mathcal{O}}}(\mathcal{R}_\lambda) = H_*^{G_{\mathcal{O}}}(\mathrm{Gr}_G^\lambda)$ is a finitely generated flat $H_{G_{\mathcal{O}}}^*(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}/W]$-module, we conclude inductively that $H_*^{G_{\mathcal{O}}}(\mathcal{R}_{\leq\lambda})$ is a finitely generated flat $\mathbb{C}[\mathfrak{t}/W]$-module. Hence it is a finitely generated projective $\mathbb{C}[\mathfrak{t}/W]$-module. Let $\mathcal{H}_{\leq\lambda}$ denote the corresponding locally free coherent sheaf on $\mathfrak{t}/W$. Then the natural morphism $\mathcal{H}_{\leq\lambda} \to j_*j^*\mathcal{H}_{\leq\lambda}$ is an isomorphism (since $\mathfrak{t}/W$ is smooth). Taking the union over all λ we obtain that the natural morphism $\mathcal{H} \to j_*j^*\mathcal{H}$ is an isomorphism, where $\mathcal{H}$ is the quasicoherent sheaf on $\mathfrak{t}/W$ localizing the $H_{G_{\mathcal{O}}}^*(\mathrm{pt}) = \mathbb{C}[\mathfrak{t}/W]$-module $H_*^{G_{\mathcal{O}}}(\mathcal{R})$.

We see that in order to identify the $\mathfrak{t}/W$-schemes $\mathcal{M}$ and $\mathrm{Spec}(H_*^{G_{\mathcal{O}}}(\mathcal{R}))$ it suffices to identify the quasicoherent $\mathfrak{t}^\bullet/W$-modules $\Pi_*\mathcal{O}_{\mathcal{M}^\bullet}$ and $j^*\mathcal{H}$: indeed, due to (1), both $\mathbb{C}[\mathcal{M}]$ and $H_*^{G_{\mathcal{O}}}(\mathcal{R})$ are subalgebras in $\mathbb{C}[\mathfrak{t}^\circ \times T^\vee]^W$, so it suffices to identify them as *subsets*. Now the desired identification is given in (1) upon restriction to $\mathfrak{t}^\circ/W \subset \mathfrak{t}^\bullet/W$, and (2) guarantees that the latter identification extends to $\mathfrak{t}^\bullet/W$. More precisely, let $\Xi^\circ = \iota_*(\mathbf{z}^*)^{-1} \circ \Xi$. Note that maps in the diagram (5.24) are W-equivariant. Hence Ξ° gives an isomorphism from $\mathbb{C}[\mathcal{M}] \otimes_{\mathbb{C}[\mathfrak{t}/W]} \mathbb{C}[\mathfrak{t}^\circ/W]$ to $H_*^{G_{\mathcal{O}}}(\mathcal{R}) \otimes_{H_G^*(\mathrm{pt})} \mathbb{C}[\mathfrak{t}^\circ/W]$. We have

$$\mathbf{z}'^*(\iota_*')^{-1}\left(\Xi^t \otimes_{\mathbb{C}[\mathfrak{t}]_t} \mathbb{C}[\mathfrak{t}^\circ]\right) = \mathbf{z}^*(\iota_*)^{-1}\left(\Xi^\circ \otimes_{\mathbb{C}[\mathfrak{t}^\circ/W]} \mathbb{C}[\mathfrak{t}^\circ]\right)$$

by the assumption in (2). By Lemma 5.25

$$\Xi^t \otimes_{\mathbb{C}[\mathfrak{t}]_t} \mathbb{C}[\mathfrak{t}^\circ] = \mathbf{z}'''^*(\iota_*''')^{-1}\left(\Xi^\circ \otimes_{\mathbb{C}[\mathfrak{t}^\circ/W]} \mathbb{C}[\mathfrak{t}^\circ]\right).$$

Note that $\mathbf{z}'''^*(\iota_*''')^{-1}$ is an isomorphism over $\mathbb{C}[\mathfrak{t}]_t$ by the localization theorem in equivariant homology groups, as $\mathcal{R}_{G',\mathbf{N}'}$ is the fixed point set of t. Since Ξ^t extends to $\mathbb{C}[\mathfrak{t}]_t$ by the assumption, $\Xi^\circ \otimes_{\mathbb{C}[\mathfrak{t}^\circ/W]} \mathbb{C}[\mathfrak{t}^\circ]$ also extends to $\mathbb{C}[\mathfrak{t}]_t$. We apply this argument to all generalized roots, we get an extension to $\mathbb{C}[\mathfrak{t}^\bullet]$. Taking the W-invariant part, we see that Ξ° extends to $\mathbb{C}[\mathfrak{t}^\bullet/W]$. $\square$

Remark 5.27. There are various ways to guarantee the condition $\Pi_*\mathcal{O}_{\mathcal{M}} \xrightarrow{\cong} j_*\Pi_*\mathcal{O}_{\mathcal{M}^\bullet}$ of Theorem 5.26. For instance, it is enough to assume that Π is flat, and $\mathcal{M}$ is Cohen-Macaulay. In effect, let $j: \mathcal{M}^\bullet \hookrightarrow \mathcal{M}$ denote also

the open embedding of the preimage of $\mathfrak{t}^\bullet/W$ in $\mathcal{M}$. We have $j_*\Pi_*\mathcal{O}_{\mathcal{M}^\bullet} = \Pi_*j_*\mathcal{O}_{\mathcal{M}^\bullet} = \Pi_*\mathcal{O}_\mathcal{M}$ because $\mathcal{M}$ is Cohen-Macaulay, and the codimension of $\mathcal{M} \setminus \mathcal{M}^\bullet$ in $\mathcal{M}$ is at least 2 because all the fibers of Π have the same dimension by flatness.

Alternatively, it is enough to assume that all the fibers of Π have the same dimension, and $\mathcal{M}$ satisfies the Serre condition S_2, e.g. $\mathcal{M}$ is normal.

6. Degeneration and its applications

We introduce a natural degeneration of $\mathcal{M}_C$ and study its applications in this section.

6(i). Filtration

Let $\overline{\mathrm{Gr}}_G^\lambda$ and $\mathcal{R}_{\leq\lambda} = \mathcal{R} \cap \pi^{-1}(\overline{\mathrm{Gr}}_G^\lambda)$ as in §2(i). In the diagram (3.1), it is known that $mqp^{-1}(\overline{\mathrm{Gr}}_G^\lambda \times \overline{\mathrm{Gr}}_G^\mu) \subset \overline{\mathrm{Gr}}_G^{\lambda+\mu}$ (see e.g., [MV07, Lemma 4.4]). Therefore we have

$$\tilde{m}\tilde{q}\tilde{p}^{-1}(\mathcal{R}_{\leq\lambda} \times \mathcal{R}_{\leq\mu}) \subset \mathcal{R}_{\leq\lambda+\mu}$$

as the diagram (3.2) is compatible with (3.1) under $\pi\colon \mathcal{T} \to \mathrm{Gr}_G$. From the definition of the convolution product, we have

$$H_*^{G_{\mathcal{O}}}(\mathcal{R}_{\leq\lambda}) * H_*^{G_{\mathcal{O}}}(\mathcal{R}_{\leq\mu}) \subset H_*^{G_{\mathcal{O}}}(\mathcal{R}_{\leq\lambda+\mu}).$$

Thus

Proposition 6.1. *$\mathcal{A}$ is a filtered algebra with respect to the filtration $\mathcal{A} = H_*^{G_{\mathcal{O}}}(\mathcal{R}) = \bigcup H_*^{G_{\mathcal{O}}}(\mathcal{R}_{\leq\lambda})$. The same is true for $\mathcal{A}_\hbar$.*

Let $\mathrm{gr}\,\mathcal{A}$ denote the associated graded algebra. It is graded by $Y^+ = Y/W$, where Y^+ is the semi-group of dominant coweights.

Thanks to Lemma 2.6(2) the associated graded algebra $\mathrm{gr}\,\mathcal{A}$ is identified with $\bigoplus H_*^{G_{\mathcal{O}}}(\mathcal{R}_\lambda)$, as an $H_{G_{\mathcal{O}}}^*(\mathrm{pt})$-module. Moreover Lemmas 2.5 and 2.6(1) imply that

$$H_*^{G_{\mathcal{O}}}(\mathcal{R}_\lambda) \cong H_{\mathrm{Stab}_G(\lambda)}^*(\mathrm{pt}) \cap [\mathcal{R}_\lambda] \cong \mathbb{C}[\mathfrak{t}]^{W_\lambda}[\mathcal{R}_\lambda],$$

where W_λ is the stabilizer of λ in the Weyl group W. We regard $[\mathcal{R}_\lambda]$ as a class of $\mathrm{gr}\,H_*^{G_{\mathcal{O}}}(\mathcal{R})$.

Let us replace G by its maximal torus T. The affine Grassmannian Gr_T for the torus T consists of discrete points parametrized by the coweight lattice Y of T. Therefore we have a direct sum decomposition $H_*^{T\circ}(\mathcal{R}_{T,\mathbf{N}_T}) = \bigoplus_{\mu \in Y} H_*^{T\circ}(\mathcal{R}_{\mu;T,\mathbf{N}_T})$, where $\mathcal{R}_{\mu;T,\mathbf{N}_T}$ is the component corresponding to a coweight μ. The above filtration for T is just the union of $H_*^{T\circ}(\mathcal{R}_{\mu;T,\mathbf{N}_T})$ such that μ is in a Weyl group orbit of a dominant coweight μ' with $\mu' \le \lambda$. Thanks to the calculation in Theorem 4.1, this is a ring having an explicit presentation by generators and relations.

In order to calculate $\mathrm{gr}\,\mathcal{A}$, let us relate $\mathrm{gr}\,\mathcal{A}$ to $\mathrm{gr}\,\mathcal{A}(T, \mathbf{N}_T)$. Recall the embedding $\iota\colon \mathcal{R}_{T,\mathbf{N}_T} \to \mathcal{R}$ appearing in Lemma 5.7. It is compatible with the filtration $\mathcal{R} = \bigcup \mathcal{R}_{\le\lambda} : \mathcal{R}_{T,\mathbf{N}_T} \cap \mathcal{R}_{\le\lambda}$ consisting of the inverse image of $W\lambda$ in $\mathcal{R}_{T,\mathbf{N}_T}$, where $W\lambda$ is the Weyl group orbit of λ considered as a subset of Gr_T. Therefore we have a homomorphism

$$\mathrm{gr}\,\iota_*\colon \ \mathrm{gr}\,\mathcal{A}(T, \mathbf{N}_T) \to \mathrm{gr}\,H_*^{T\circ}(\mathcal{R}) = \mathrm{gr}\,\mathcal{A}(G, \mathbf{N}) \otimes_{H_G^*(\mathrm{pt})} H_T^*(\mathrm{pt}).$$

Thanks to Lemma 5.17, it is a graded algebra homomorphism, which becomes an isomorphism over $\mathfrak{t}^\circ$. In particular, it is injective.

Let us compute structure constants of the multiplication in $[\mathcal{R}_\lambda]$, via $\mathrm{gr}\,\mathcal{A}(T, \mathbf{N}_T)$. Recall r^λ is the fundamental class of the fiber of $\mathcal{R}_{T,\mathbf{N}_T} \to \mathrm{Gr}_T$ at a point corresponding to a (not necessarily dominant) coweight λ. (See the proof of Theorem 4.1.) Let $a_{\lambda,\mu} \in \mathbb{C}[\mathfrak{t}]$ denote the coefficient given by $r^\lambda r^\mu = a_{\lambda,\mu} r^{\lambda+\mu}$. See (4.2) for the explicit form.

Proposition 6.2. *Let λ, μ be dominant coweights. Let f, $g \in \mathbb{C}[\mathfrak{t}]^{W_\lambda}$, $\mathbb{C}[\mathfrak{t}]^{W_\mu}$ respectively. We have the following identity in the associated graded ring $\mathrm{gr}\,\mathcal{A}$:*

$$f[\mathcal{R}_\lambda] * g[\mathcal{R}_\mu] = a_{\lambda,\mu} fg[\mathcal{R}_{\lambda+\mu}].$$

The same is true for $\mathcal{A}_\hbar$.

Proof. We have

$$(6.3) \qquad (\mathrm{gr}\,\iota_*)^{-1} f[\mathcal{R}_\lambda] = \sum_{\lambda' \in W\lambda} \frac{wf \times r^{\lambda'}}{e(T_{\lambda'}\mathrm{Gr}_G^\lambda)},$$

where $T_{\lambda'}\mathrm{Gr}_G^\lambda$ is the tangent space of Gr_G^λ at the point $z^{\lambda'}$ with $\lambda' \in W\lambda$, and $e(T_{\lambda'}\mathrm{Gr}_G^\lambda)$ is its equivariant Euler class. Furthermore w for wf is given so that $\lambda' = w\lambda$. Then wf is independent of the choice as f is invariant under W_λ. Since $\mathrm{gr}\,\iota_*$ is an algebra homomorphism, we can calculate $f[\mathcal{R}_\lambda] * g[\mathcal{R}_\mu]$

from the right hand side. Since it is enough to compute the coefficient of $r^{\lambda+\mu}$, we do not need to worry other terms than r^λ, r^μ. Therefore we obtain

$$f[\mathcal{R}_\lambda] * g[\mathcal{R}_\mu] = \frac{e(T_{\lambda+\mu}\mathrm{Gr}_G^{\lambda+\mu})}{e(T_\lambda\mathrm{Gr}_G^\lambda)e(T_\mu\mathrm{Gr}_G^\mu)}a_{\lambda,\mu}fg[\mathcal{R}_{\lambda+\mu}].$$

Now we use $e(T_\lambda\mathrm{Gr}_G^\lambda) = \prod_{\alpha\in\Delta^+}\alpha^{\langle\lambda,\alpha\rangle}$, where α is considered as an element of $\mathbb{C}[\mathfrak{t}]$ (cf. the proof of Lemma 2.5). Therefore $e(T_\lambda\mathrm{Gr}_G^\lambda)e(T_\mu\mathrm{Gr}_G^\mu)$ cancels with $e(T_{\lambda+\mu}\mathrm{Gr}_G^{\lambda+\mu})$ and we get the assertion. $\qquad\square$

Remark 6.4. Let us check that $a_{\lambda,\mu}fg \in \mathbb{C}[\mathfrak{t}]^{W_{\lambda+\mu}}$. From the description of $a_{\lambda,\mu}$ (see (4.2)), it is clear that $a_{\lambda,\mu} \in \mathbb{C}[\mathfrak{t}]^W$. Note also that W_λ is the subgroup of W generated by simple reflections s_i such that the corresponding simple roots α_i are perpendicular to λ. Therefore we have $W_{\lambda+\mu} = W_\lambda \cap W_\mu$, and hence $fg \in \mathbb{C}[\mathfrak{t}]^{W_{\lambda+\mu}}$.

Remark 6.5. In [BDG15, §4.3], a monopole operator $M_{A,p} \in \mathbb{C}[\mathcal{M}_C]$ corresponding to a cocharacter A and an W_A-invariant polynomial p is considered. Here W_A is the Weyl group of $\mathrm{Stab}_G(A)$. Moreover a formula for the product $M_{A,p}M_{A',p'}$ is proposed. (See [BDG15, (4.16,17)].) They satisfy the triangular property, which is related to our filtration. Therefore $M_{A,p}$ could be related to our $f[\mathcal{R}_\lambda]$ ($f \in \mathbb{C}[\mathfrak{t}]^{W_\lambda}$) under $A \leftrightarrow \lambda$, $p \leftrightarrow f$. However our $f[\mathcal{R}_\lambda]$ lives in gr $\mathcal{A}$. It is not clear for us how to lift $f[\mathcal{R}_\lambda]$ to $\mathbb{C}[\mathcal{M}_C]$ *canonically*. Also it is not clear for us how to define the equivariant integration in [BDG15, (4.16,17)] rigorously.

6(ii). Closed $G_\mathcal{O}$-orbits

The discussion in the previous subsection was about the associated graded, but we can also say something if Gr_G^λ is a closed $G_\mathcal{O}$-orbit.

First note that the fundamental class $[\mathcal{R}_\lambda]$ is well-defined in $H_*^{G_\mathcal{O}}(\mathcal{R})$ as $\mathcal{R}_{<\lambda} = \emptyset$. Let ι_* as in Lemma 5.7, and $r^{\lambda'}$ denote the fundamental class of the fiber of $\mathcal{R}_{T,\mathbf{N}_T} \to \mathrm{Gr}_T$ at a coweight λ' as above. Then (6.3) remains true:

Proposition 6.6. *Let λ be a dominant weight such that Gr_G^λ is a closed $G_\mathcal{O}$-orbit. Let $f \in \mathbb{C}[\mathfrak{t}]^{W_\lambda}$. Then*

$$(\iota_*)^{-1}f[\mathcal{R}_\lambda] = \sum_{\lambda'\in W\lambda}\frac{wf \times r^{\lambda'}}{e(T_{\lambda'}\mathrm{Gr}_G^\lambda)}.$$

Recall that we have explicit structure constants for $r^{\lambda'}$ (Theorem 4.1) and ι_* is an algebra homomorphism (Lemma 5.10). Therefore this proposition can be used to compute multiplication of elements of forms $f[\mathcal{R}_\lambda]$ with Gr_G^λ closed.

Remark 6.7. It has been noted that monopole operators $M_{A,p}$ have explicit presentation when A is minuscule in [BDG15, §4.3]. This observation is compatible with the above proposition, i.e., $f[\mathcal{R}_\lambda]$ has well-defined lift when λ is minuscule, in view of Remark 6.5. As is noted [BDG15], there are many examples such that minuscule monopole operators generate $\mathcal{A}$, for example when G is a product of GL and PGL like a quiver gauge theory. See the proof of Lemma 6.9(2) below, for example.

6(iii). Finite generation

Proposition 6.8. $\mathcal{A}$ and $\mathcal{A}_\hbar$ are finitely generated. They are noetherian.

Proof. As for finite generation, it is enough to check that for $\mathcal{A}$ since $\mathcal{A}_\hbar$ is a flat deformation of $\mathcal{A}$. Since $\mathcal{A}$ is commutative, it is noetherian if it is finitely generated. Then $\mathcal{A}_\hbar$ is also noetherian. Thus it is enough to check that $\mathcal{A}$ is finitely generated. Furthermore, it is sufficient to show that $\mathrm{gr}\,\mathcal{A}$ is finitely generated.

We consider generalized (closed) Weyl chambers given by complements of generalized root hyperplanes. Our λ, μ are in the dominant Weyl chamber in the usual sense, but the dominant Weyl chamber is further decomposed into union of generalized Weyl chambers as there might be weights which are not roots.

Now each generalized chamber is a rational polyhedral cone, hence its intersection with the coweight lattice is a finitely generated semigroup. We take finitely many $[\mathcal{R}_\lambda]$ so that λ is a generator of the semigroup. If we multiply $[\mathcal{R}_\lambda]$, $[\mathcal{R}_\mu]$ when λ, μ are in the same generalized Weyl chamber, $a_{\lambda\mu}$ is 1 from its definition. Therefore $[\mathcal{R}_\lambda][\mathcal{R}_\mu] = [\mathcal{R}_{\lambda+\mu}]$. Therefore $[\mathcal{R}_\lambda]$ as above for each generalized chamber, together with generators of $\mathbb{C}[\mathfrak{t}]$ generate $\mathrm{gr}\,\mathcal{A}$. □

6(iv). SL(2) and PGL(2) cases

We determine $\mathcal{M}_C$ when $G = \mathrm{SL}(2)$ or $\mathrm{PGL}(2)$ in this subsection.[7]

Lemma 6.9. _____

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[7]This result was taught by Amihay Hanany as a result of physical intuition. The third named author thanks him for his explanation.

(1) *Suppose $G = \mathrm{SL}(2)$. Then $\mathcal{M}_C(G, \mathbf{N})$ is a hypersurface in $\mathbb{C}^3$ of the form $\xi^2 = \delta\eta^2 - \delta^{N-1}$ $(N \geq 1)$ or $\xi^2 = \delta\eta^2 + \eta$ $(N = 0)$, where $N = \sum_\mu |\langle \chi, \lambda_0 \rangle| \dim \mathbf{N}(\chi)/2$. Here λ_0 is the generator of the coweight lattice, which is dominant.*

(2) *Suppose $G = \mathrm{PGL}(2)$. Then $\mathcal{M}_C(G, \mathbf{N})$ is also $\xi^2 = \delta\eta^2 - \delta^{N-1}$ where $N = \sum_\mu |\langle \chi, \lambda_0 \rangle| \dim \mathbf{N}(\chi)/2 + 1$. Furthermore we have $\mathcal{R}_{\leq \lambda_0} = \mathcal{R}_{\lambda_0}$ in this case, and generators are given by $\eta = [\mathcal{R}_{\lambda_0}]$, $\xi = t[\mathcal{R}_{\lambda_0}]$, $\delta = t^2 \in H^4_{\mathbb{C}^\times}(\mathrm{pt})^{\{\pm 1\}}$, up to multiplicative constants, where $H^*_{\mathbb{C}^\times}(\mathrm{pt}) = \mathbb{C}[t]$. Here λ_0 is again the generator of the coweight lattice, which is dominant. (It is the half of λ_0 in (1).).*

Proof. Let us first consider both $G = \mathrm{SL}(2)$ and $\mathrm{PGL}(2)$ together.

We use $\mathrm{gr}\,\mathcal{A}$.

Let us identify the torus T with $\mathbb{C}^\times$ and the coweight lattice with $\mathbb{Z}$ so that dominant coweights are $\mathbb{Z}_{\geq 0}$. (Hence λ_0 in the statement is 1.) Since $\lambda, \mu \in \mathbb{Z}_{>0}$ are equal up to multiple of $\mathbb{Q}_{>0}$, $\langle \chi, \lambda \rangle$ and $\langle \chi, \mu \rangle$ always have the same sign for any weight χ. Therefore $a_{\lambda,\mu}$ in Proposition 6.2 is 1. Moreover $W_\lambda = \{\pm 1\}$ if $\lambda = 0$, $= \{1\}$ if $\lambda > 0$. Therefore $\mathrm{gr}\,\mathcal{A}$ is generated by $\delta = t^2 \in H^4_{\mathbb{C}^\times}(\mathrm{pt})^{\{\pm 1\}}$, $\eta = [\mathcal{R}_1]$, $\xi = t[\mathcal{R}_1]$, where t is a generator of $\mathbb{C}[t] \cong H^*_{\mathbb{C}^\times}(\mathrm{pt})$. Moreover the relation is

$$(6.10) \qquad \xi^2 = (t[\mathcal{R}_1])^2 = t^2[\mathcal{R}_2] = \delta\eta^2.$$

The singularities of $\mathrm{Spec}(\mathrm{gr}\,\mathcal{A})$ is at $\xi = \eta = 0$ and arbitrary δ. (In particular $\mathrm{Spec}(\mathrm{gr}\,\mathcal{A})$ is *not* normal.)

Since $\mathrm{gr}\,\mathcal{A}$ is a degeneration of $\mathcal{A}$, δ, η, ξ are generators of $\mathcal{A}$, hence $\mathcal{M}_C$ is also a hypersurface in $\mathbb{C}^3$.

Note that (6.10) is true modulo $H^{G_{\mathcal{O}}}_*(\mathcal{R}_{\leq 1})$ and the defining equation must be homogeneous with $\deg \xi = \deg \eta + 1$, $\deg \delta = 2$. The only possibility is

$$\xi^2 - \delta\eta^2 = a\delta^m + b\delta^n\xi \text{ or } + b\delta^n\eta$$

for some $m, n \in \mathbb{Z}_{\geq 0}$, $a, b \in \mathbb{C}$. ($m = \deg \eta + 1$, $n = (\deg \eta + 1)/2$ or $(\deg \eta + 2)/2$.)

Consider the first case $\xi^2 - \delta\eta^2 = a\delta^m + b\delta^n\xi$. We have $m = 2n$. We set $\xi' \overset{\text{def.}}{=} \xi - b\delta^n/2$. Then $(\xi')^2 - \delta\eta^2 = (a - b^2/4)\delta^m$. Note that $a = b^2/4$ is not possible, as we know that $\mathcal{M}_C$ is nonsingular for $\delta \neq 0$ by Corollary 5.21 as $0 \neq \delta \in \mathfrak{t}^\circ$. Therefore by rescaling δ and η, we get the equation $(\xi')^2 - \delta\eta^2 = \delta^m$.

The other case $\xi^2 - \delta\eta^2 = a\delta^m + b\delta^n\eta$ is the same if $n \neq 0$. If $n = 0$, $m = -1$. Therefore $a\delta^m$ cannot appear. Hence we have $\xi^2 - \delta\eta^2 = \eta$ by rescaling.

We now suppose $G = \mathrm{PGL}(2)$. In this case $\mathrm{Gr}_G^{\lambda_0}$ is a closed $G_{\mathcal{O}}$-orbit for $\lambda_0 = 1$. Hence Proposition 6.6 is applicable for η, ξ. We have $\mathrm{Gr}_G^{\lambda_0} \cong \mathbb{P}^1$ and $W\lambda_0 = \{\pm\lambda_0\}$ is the north and south poles. We have

$$(\iota_*)^{-1}(\eta) = \frac{r^{\lambda_0} - r^{-\lambda_0}}{t}, \qquad (\iota_*)^{-1}(\xi) = r^{\lambda_0} + r^{-\lambda_0}.$$

Therefore

$$(\iota_*)^{-1}(\xi^2 - \delta\eta^2) = 4 r^{\lambda_0} r^{-\lambda_0} = 4 \prod_{\mu} (\langle \mu, \lambda_0 \rangle t)^{|\langle \mu, \lambda_0 \rangle|},$$

where the last equality is by (4.2). Note that the power of t is even as weights appear in pairs μ, $-\mu$. Thus the right hand side is a power of δ. After rescaling ξ, δ, η, we get an equation $\xi^2 - \delta\eta^2 = \delta^m$.

Finally the formula of N is given by computing the degree of $\eta = [\mathcal{R}_1]$. The rank of $\mathcal{T}/\mathcal{R}$ over Gr_G^1 is given by Lemma 2.2, and $\dim \mathrm{Gr}_G^1 = 1$ for $\mathrm{PGL}(2)$, 2 for $\mathrm{SL}(2)$. (For $\mathrm{PGL}(2)$, the above computation also gives the formula of $m = N - 1$.) $\square$

Remark 6.11. The hypersurface $\xi^2 = \delta\eta^2 - \delta^{N-1}$ is a simple singularity of type D_N if $N \geq 4$. The cases $\xi^2 = \delta\eta^2 + \eta$ ('type D_0') and $N = 1$ are nonsingular. $N = 2$ has two A_1 singularities at $\xi = \delta = 0$, $\eta = \pm 1$. $N = 3$ is isomorphic to the A_3-singularity. (The natural hyper-Kähler metrics on D_3 and A_3 are different.) Compare with Example 3.21.

6(v). Normality

Proposition 6.12. $\mathcal{M}_C(G, \mathbf{N})$ *is a normal variety.*

The proof occupies this subsection. We start with the following lemma.

Lemma 6.13. *Let X be an affine scheme of finite type over a field $\Bbbk$ and let U be an open subset of X such that the complement has dimension ≥ 2. Assume that a) U is normal, b) any regular function on U extends to X. Then X is normal.*

Proof. We use Serre's criterion (R_1) and (S_2). By a) U satisfies both (R_1) and (S_2).

By (R_1) for U and $\mathrm{codim}(X \setminus U) \geq 2$, X satisfies the condition (R_1). The condition (S_2) is guaranteed by (S_2) for U and b), as (S_2) is equivalent

to that any regular function on V extends to X for arbitrary open subscheme $V \subset X$ with $\mathrm{codim}(X \setminus V) \geq 2$. $\qquad\square$

We take $X = \mathcal{M}_C(G, \mathbf{N})$, $U = \varpi^{-1}(\mathfrak{t}^\bullet/W)$. The condition b) is satisfied thanks to the first part of the proof of Theorem 5.26. We also know ϖ is flat (Lemma 5.3) and X is irreducible (Corollary 5.22). Therefore fibers of ϖ all have the same dimension, hence the complement of U has codimension 2. Therefore it is enough to show that U is normal.

By the localization theorem in equivariant cohomology groups, we have an isomorphism

$$H_*^{Z_G(t)\circ}(\mathcal{R}_{Z_G(t),\mathbf{N}^t}) \otimes_{H_{Z_G(t)}^*(\mathrm{pt})} \mathbb{C}[\mathfrak{t}]_t^{W_t} \cong H_*^{G\circ}(\mathcal{R}_{G,\mathbf{N}}) \otimes_{H_G^*(\mathrm{pt})} \mathbb{C}[\mathfrak{t}]_t^{W_t}$$

for $t \in \mathfrak{t}$. (Recall §5, especially (5.24).) Here W_t is the Weyl group of $Z_G(t)$, which is the subgroup of W fixing t. Therefore it is enough to show the normality of $\mathcal{M}_C(Z_G(t), \mathbf{N}^t)$ for each $t \in \mathfrak{t}^\bullet$. If $t \in \mathfrak{t}^\circ$, we have $\mathcal{M}_C(Z_G(t), \mathbf{N}^t) \cong T^* T^\vee$. So it is smooth. Therefore we may assume $t \in \mathfrak{t}^\bullet \setminus \mathfrak{t}^\circ$. By §5(i), there is a unique generalized root α such that $\langle t, \alpha \rangle = 0$.

As remarked at the beginning of §5, $Z_G(t)$ has semisimple rank at most 1, and we know the answer for $\mathrm{SL}(2)$, $\mathrm{PGL}(2)$ and a torus T. The remaining task is a reduction to these cases.

Proof of Proposition 6.12. The same (or simpler) argument as in Lemma 6.9 shows that $\mathcal{M}_C(G, \mathbf{N})$ for $G = \mathbb{C}^\times$ is a hypersurface $xy = w^N$ for some $N = 0, 1, 2, \dots$.

Slightly more generally, suppose that α is a generalized root of type (I). Then $Z_G(t) = T$, $\mathbf{N}^t = \mathbf{N}^T \oplus \bigoplus_{m \in \mathbb{Z}} \mathbf{N}(m\alpha)$. Let us consider $\mathcal{A}(Z_G(t), \mathbf{N}^t) = \mathcal{A}(T, \bigoplus_{m \in \mathbb{Z}} \mathbf{N}(m\alpha))$. Let $Y(T)$ be the coweight lattice of T. Since α is a weight of T, we have a homomorphism $\Phi \colon Y(T) \to \mathbb{Z}$ given by the pairing with α. Let us consider the kernel and image

$$0 \to \mathrm{Ker}\,\Phi \to Y(T) \to \mathrm{Im}\,\Phi \to 0.$$

Since $\mathrm{Ker}\,\Phi$ and $\mathrm{Im}\,\Phi$ are both free, this exact sequence splits, hence $Y(T) \cong \mathrm{Ker}\,\Phi \oplus \mathrm{Im}\,\Phi$. Let us take $\lambda_g \in \mathrm{Im}\,\Phi \cong \mathbb{Z}$, a generator. We suppose $\langle \lambda_g, \alpha \rangle > 0$. Then by Theorem 4.1, $\mathcal{A}(T, \bigoplus_{m \in \mathbb{Z}} \mathbf{N}(m\alpha))$ is generated by $\mathbb{C}[\mathfrak{t}]$, r^λ ($\lambda \in \mathrm{Ker}\,\Phi$), $r^{\pm\lambda_g}$ with the relation

$$r^\lambda r^\mu = r^{\lambda+\mu} \quad (\lambda, \mu \in \mathrm{Ker}\,\Phi), \qquad r^{\lambda_g} r^{-\lambda_g} = \prod_{m \in \mathbb{Z}} (m\alpha)^{|m|\langle \lambda_g, \alpha \rangle \dim \mathbf{N}(m\alpha)},$$

where α is regarded as an element in $\mathfrak{t}^*$. Therefore $\mathcal{M}_C(Z_G(t), \mathbf{N}^t)$ is a product of T^*T' for one dimensional lower torus T' and a simple singularity $xy = w^N$ of type A for some $N = 1, 2, \ldots$.

If α is a type (II) generalized root, $Z_G(t)$ has semisimple rank 1. Let us denote by Z the neutral connected component of the center of $Z_G(t)$. It acts trivially on $\mathbf{N}^t$ since its Lie algebra $\mathrm{Lie}(Z) \subset \mathfrak{t}$ is the kernel of α in the Cartan subalgebra, but the weights of $\mathbf{N}^t$ are multiples of α. Hence the action of $Z_G(t)$ on $\mathbf{N}^t$ factors through $H := Z_G(t)/Z$. Note that H is isomorphic to $\mathrm{SL}(2)$ or $\mathrm{PGL}(2)$ (recall that $Z_G(t)$ is connected). Let $D \subset Z_G(t)$ be the derived subgroup, and let $T' := Z_G(t)/D$ be the quotient torus. The kernel of the diagonal morphism $Z_G(t) \to T' \times H$ is a finite abelian subgroup Γ (in fact, Γ is either $\{\pm 1\}$ or trivial), so that we have an exact sequence $1 \to \Gamma \to Z_G(t) \to T' \times H \to 1$. As we have just seen, the representation of $Z_G(t)$ in $\mathbf{N}^t$ factors through its quotient $Z_G(t) \to T' \times H \to H$. Hence $H^{Z_G(t)_\circ}(\mathcal{R}_{Z_G(t),\mathbf{N}^t}) = H^{(T' \times H)_\circ}(\mathcal{R}_{T' \times H,\mathbf{N}^t})^{\Gamma^\wedge} = \mathbb{C}[T^*T'^\vee \times \mathcal{M}_C(H,\mathbf{N}^t)]^{\Gamma^\wedge}$ by §3(vii)3(vii),3(vii). Since $\mathcal{M}_C(H,\mathbf{N}^t)$ is normal according to Lemma 6.9, $(T^*T'^\vee \times \mathcal{M}_C(H,\mathbf{N}^t))/\Gamma^\wedge$ is normal as well. $\square$

6(vi). Adjoint matters

The purpose of this subsection is to prove the following result, mentioned in §3(x)(b).

Proposition 6.14. *For a reductive group G and its adjoint representation $\mathfrak{g}$, the birational isomorphism $\mathbf{z}^*\iota_*^{-1} \colon \mathcal{M}_C(G,\mathfrak{g})|_{\Phi^{-1}(\mathfrak{t}^\circ/W)} \simeq (\mathfrak{t}^\circ \times T^\vee)/W$ of Corollary 5.21 extends to a biregular isomorphism $\mathcal{M}_C(G,\mathfrak{g}) \xrightarrow{\sim} (\mathfrak{t} \times T^\vee)/W$.*

Proof. We use the criterion in Theorem 5.26.[8] In this case generalized roots are nothing but usual roots. For $t \in \mathfrak{t}^\bullet \setminus \mathfrak{t}^\circ$, there is a single root α with $\langle \alpha, t \rangle = 0$. By the commutativity of (5.24), Ξ^t is also given by Corollary 5.21, but for $Z_G(t)$. Therefore it is enough to check the assertion for $Z_G(t)$. By the argument in the last part of the proof in Proposition 6.12, we can replace $Z_G(t)$ by $\mathrm{PGL}(2)$. ($\mathrm{SL}(2)$ can be replaced by $\mathrm{PGL}(2)$, as we are considering the adjoint representation.)

For $G = \mathrm{PGL}(2)$, let us use the computation in the proof of Lemma 6.9(2). Thanks to §4(vi), we have $\mathbf{z}^*(\iota_*)^{-1}(\eta) = r^{\lambda_0} + r^{-\lambda_0}$, $\mathbf{z}^*(\iota_*)^{-1}(\xi) = t(r^{\lambda_0} - r^{-\lambda_0})$, where $r^{\pm\lambda_0}$ is the fundamental class of the point $\pm\lambda_0 \ (= \pm 1)$ in Gr_T.

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[8]The third named author thanks Ryosuke Kodera for his suggestion to give this proof.

Now we see that $\mathbf{z}^* \iota_*^{-1}$ is an isomorphism, as r^{λ_0} and t are coordinates of $T^\vee$ and $\mathfrak{t}$ respectively. $\qquad\square$

6(vii). Symplectic form

Proposition 6.15. *The Poisson structure is symplectic on the smooth locus of $\mathcal{M}_C$.*

Proof. By Corollary 5.21, we already know the assertion over $\varpi^{-1}(\mathfrak{t}^\circ/W)$. By Hartogs theorem, it is enough to check that the symplectic form extends and is nondegenerate up to codimension 2. As in the proof of Proposition 6.12, we check it for $\varpi^{-1}(\mathfrak{t}^\bullet/W)$, then it is enough to assume $G = \mathbb{C}^\times$, SL(2) or PGL(2).

Let us consider $G = \mathbb{C}^\times$. Then $\mathcal{M}_C$ is a hypersurface $xy = w^N$ in $\mathbb{C}^3$ ($N = 0, 1, \dots$). The birational isomorphism $\mathbf{z}^*(\iota_*)^{-1}\colon \mathcal{M}_C \dashrightarrow^{\approx} T^* \mathbb{C}^\times = \mathbb{C} \times \mathbb{C}^\times$ in Corollary 5.21 is given by $(x, y, w) \mapsto (w, x)$ defined over $w \neq 0$. Moreover the symplectic structure on $T^* \mathbb{C}^\times$ is $x^{-1} dx \wedge dw$. We can rewrite it as $-y^{-1} dy \wedge dw$. Hence it is a well-defined symplectic form over $\{x \neq 0\} \cup \{y \neq 0\} = \mathcal{M}_C \setminus \{x = y = w = 0\}$. If $N = 0$, $\mathcal{M}_C = \mathbb{C} \times \mathbb{C}^*$, i.e., it is well-defined everywhere. If $N = 1$, we have $w = xy$, hence $x^{-1} dx \wedge dw = dx \wedge dy$. Hence it is also well-defined and non-degenerate over the whole $\mathcal{M}_C$. (It is also a consequence of Hartogs theorem.) For general N, it is a symplectic form on $\mathcal{M}_C \cong \mathbb{C}^2/(\mathbb{Z}/N)$, descending from the standard one on $\mathbb{C}^2$, divided by N.

Let us next consider the case $G = \text{PGL}(2)$. Let $\xi = t[\mathcal{R}_\lambda]$, $\eta = [\mathcal{R}_\lambda]$ as in the proof of Lemma 6.9. We use $\mathbf{z}^*(\iota_*)^{-1}\colon \mathcal{M}_C \dashrightarrow^{\approx} (\mathbb{C} \times \mathbb{C}^\times)/(\mathbb{Z}/2)$. Let t, $a^\pm$ be coordinates of $\mathbb{C}$ and $\mathbb{C}^\times$ respectively. The Weyl group action is $t \leftrightarrow -t$, $a \leftrightarrow a^{-1}$. First note that δ is sent to t^2. Recall $(\iota_*)^{-1}(\eta) = t^{-1}(r^\lambda - r^{-\lambda})$, $(\iota_*)^{-1}(\xi) = r^\lambda + r^{-\lambda}$ in the proof of Lemma 6.9(2). Thanks to §4(vi), we have $\mathbf{z}^*(\iota_*)^{-1}(\eta) = t^{N-2}(a \pm a^{-1})$, $\mathbf{z}^*(\iota_*)^{-1}(\xi) = t^{N-1}(a \mp a^{-1})$ up to the same multiplicative constant. Here $\pm$ is determined according to the parity of N so that those are invariant under the Weyl group action. Let us rescale ξ, η so that these formulas are true without ambiguity. (Hence the defining equation is $\xi^2 = \delta \eta^2 \mp 4 \delta^{N-1}$.)

The standard symplectic form on $\mathbb{C} \times \mathbb{C}^\times$ is $a^{-1} da \wedge dt$, and descends to $\mathbb{C} \times \mathbb{C}^\times/(\mathbb{Z}/2)$. Its pull-back is

$$\frac{1}{2}\xi^{-1} d\eta \wedge d\delta \quad \text{over } \xi \neq 0, \qquad \frac{1}{2}\delta^{-1}\eta^{-1} d\xi \wedge d\delta \quad \text{over } \delta\eta \neq 0,$$
$$\left(\eta^2 \mp 4(N-1)\delta^{N-2}\right)^{-1} d\eta \wedge d\xi \quad \text{over } \eta^2 \mp 4(N-1)\delta^{N-2} \neq 0.$$

Therefore the pull-back is well-defined and nondegenerate over $\mathcal{M}_C \setminus \{\xi = \delta\eta = \eta^2 \mp 4(N-1)\delta^{N-2} = 0\}$. The complement is empty if $N = 1$; $\xi = \delta = 0$, $\eta = 2$, -2 if $N = 2$; and $\xi = \delta = \eta = 0$ if $N \geq 3$. It is exactly the singular locus of $\mathcal{M}_C$. In fact, it is the descent of the standard symplectic form on $\mathbb{C}^2$ up to constant if $N \geq 3$.

For $G = \mathrm{SL}(2)$, lifts of $[\mathcal{R}_\lambda]$, $t[\mathcal{R}_\lambda]$ are well-defined only up to $\mathbb{C}\delta^m$ for appropriate m. (In fact, either of two is well-defined by the degree reason.) By adjusting the ambiguity, we take η, ξ so that $\mathbf{z}^*(\iota_*)^{-1}(\eta) = t^{N-2}(a \pm a^{-1})$, $\mathbf{z}^*(\iota_*)^{-1}(\xi) = t^{N-1}(a \mp a^{-1})$ are true. Then the remaining argument is the same. The case $N = 0$ is exceptional. In this case, there is no ambiguity of $\mathbb{C}\delta^m$ by degree reason. We determine η as $\mathbf{z}^*(\iota_*)^{-1}(\eta) = t^{-2}(a + a^{-1} - 2)$ so that the defining equation $\xi^2 = \delta\eta^2 + 4\eta$ has no negative powers of δ. Then the symplectic form is $\xi^{-1}d\eta \wedge d\delta/2$ over $\xi \neq 0$, $(\delta\eta + 2)^{-1}d\xi \wedge d\delta/2$ over $\delta\eta + 2 \neq 0$ and $\eta^{-2}d\eta \wedge d\xi$ over $\eta \neq 0$. Since $\xi = \eta = 0$ and $\delta\eta + 2 = 0$ cannot happen simultaneously, we conclude that the symplectic form is well-defined and non-degenerate on the whole $\mathcal{M}_C$. (In this case $\mathcal{M}_C$ is nonsingular.) $\qquad\square$

6(viii). Changing a summand to its dual

Let us combine §4(v) with Theorem 5.26 to show that $\mathcal{M}_C(G, \mathbf{N})$ depends only on $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$. Let $\underline{\mathbf{N}}$ be a summand of $\mathbf{N}$, and let $\mathbf{N}'$ be the representation of G obtained by replacing $\underline{\mathbf{N}}$ by its dual $\underline{\mathbf{N}}^*$.

Let us define an automorphism of $H_*^{T_{\mathbb{O}}}(\mathrm{Gr}_T) \cong \mathbb{C}[\mathfrak{t} \times T^\vee]$ by

$$r^\lambda \mapsto (-1)^{\sum_\chi \max(\chi(\lambda), 0) \dim \underline{\mathbf{N}}(\chi)} r^\lambda,$$

where $\underline{\mathbf{N}}(\chi)$ is the weight χ subspace of $\underline{\mathbf{N}}$. This definition is based on the definition in §4(v). This is W-equivariant, hence induces an automorphism of $\mathfrak{t} \times T^\vee/W$. Composing with birational isomorphisms $\mathcal{M}_C(G, \mathbf{N}) \dashrightarrow \mathfrak{t} \times T^\vee/W$, $\mathcal{M}_C(G, \mathbf{N}') \dashrightarrow \mathfrak{t} \times T^\vee/W$ in Corollary 5.21, we consider the above as a birational isomorphism $\mathcal{M}_C(G, \mathbf{N}) \dashrightarrow \mathcal{M}_C(G, \mathbf{N}')$. We claim that it extends to a biregular isomorphism $\mathcal{M}_C(G, \mathbf{N}) \xrightarrow{\cong} \mathcal{M}_C(G, \mathbf{N}')$. By Theorem 5.26 and a similar argument in the proof of Proposition 6.12, we may assume G is either torus, $\mathrm{PGL}(2)$ or $\mathrm{SL}(2)$.

By §4(v) the assertion is true when G is torus.

Suppose $G = \mathrm{PGL}(2)$. We may assume $\underline{\mathbf{N}} = S^k(\mathbb{C}^2)$ with k even. Then $\chi(\lambda_0)$ is $k/2$, $k/2 - 1, \ldots, -k/2$. Therefore $\sum_\chi \max(\chi(\pm\lambda^0), 0) \dim \underline{\mathbf{N}}(\chi) = k(k+2)/8$. Since $(\iota_*)^{-1}\xi = r^{\lambda_0} + r^{-\lambda_0}$, $(\iota_*)^{-1}\eta = (r^{\lambda_0} - r^{-\lambda_0})/t$, $\delta = t^2$ (see

the proof of Lemma 6.9), (ξ, η, δ) is mapped to $(\pm\xi, \pm\eta, \delta)$, hence the equation $\xi^2 = \delta\eta^2 - c\delta^{N-1}$ $(c \in \mathbb{C})$ is preserved.

Suppose $G = \mathrm{SL}(2)$ and $\underline{\mathbf{N}} = S^k(\mathbb{C}^2)$. Then $\chi(\lambda_0)$ is k, $k - 2, \dots, -k$. Therefore $\sum_\chi \max(\chi(\pm\lambda^0), 0) \dim \underline{\mathbf{N}}(\chi) = k(k+2)/4$ if k is even, $(k+1)^2/4$ if k is odd. As in the proof of Proposition 6.15, we adjust ξ, η by $\mathbb{C}\delta^m$ so that $(\iota_*)^{-1}\xi = (r^{\lambda_0} - r^{-\lambda_0})/t$, $(\iota_*)^{-1}\eta = (r^{\lambda_0} + r^{-\lambda_0})/t^2$. Thus the birational isomorphism is given by $(\xi, \eta, \delta) \mapsto (\pm\xi, \pm\eta, \delta)$, hence the equation $\xi^2 = \delta\eta^2 - c\delta^{N-1}$ $(c \in \mathbb{C})$ is preserved.

6(ix). Another degeneration

We have a $\mathbb{C}^\times$-action on $\mathcal{R}$ induced from the dilatation on $\mathbf{N}$. Let us denote the variable for the equivariant cohomology of this $\mathbb{C}^\times$ by $\mathbf{t}$, i.e., $H_{\mathbb{C}^\times}^*(\mathrm{pt}) = \mathbb{C}[\mathbf{t}]$. Let us consider the embedding

$$\mathbf{z}^* \colon H_*^{\mathbb{C}^\times \times G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}) \to H_*^{\mathbb{C}^\times \times G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G) \cong H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)[\mathbf{t}]$$

(where the second factor $\mathbb{C}^\times$ stands for the loop rotation). We extend it to $H_*^{\mathbb{C}^\times \times G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}) \otimes_{\mathbb{C}[\mathbf{t}]} \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}] \to H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)[\mathbf{t}, \mathbf{t}^{-1}]$. Let $\mathcal{L} \equiv \mathcal{L}_{G,\mathbf{N}}$ be the pull-back of the $\mathbb{C}[\mathbf{t}^{-1}]$-lattice $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)[\mathbf{t}^{-1}]$ by $\mathbf{z}^*$. We have the induced injective ring homomorphism

$$(6.16) \qquad\qquad \overline{\mathbf{z}}^* \colon \mathcal{L}/\mathbf{t}^{-1}\mathcal{L} \to H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G).$$

Proposition 6.17. $\overline{\mathbf{z}}^* \colon \mathcal{L}/\mathbf{t}^{-1}\mathcal{L} \xrightarrow{\sim} H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ *is an isomorphism.*

Proof. Recall the multifiltrations on $H_*^{\mathbb{C}^\times \times G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R})$, $H_*^{\mathbb{C}^\times \times G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ introduced in §6(i). They induce the multifiltrations on $H_*^{\mathbb{C}^\times \times G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathcal{R}) \otimes_{\mathbb{C}[\mathbf{t}]} \mathbb{C}[\mathbf{t}, \mathbf{t}^{-1}]$, $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)[\mathbf{t}, \mathbf{t}^{-1}]$, and $\mathbf{z}^*$ is compatible with the filtrations. Hence it induces a ring homomorphism

$$(6.18) \qquad \mathrm{gr}\,\overline{\mathbf{z}}^* \colon \mathrm{gr}(\mathcal{L}/\mathbf{t}^{-1}\mathcal{L}) = (\mathrm{gr}\,\mathcal{L})/\mathbf{t}^{-1}(\mathrm{gr}\,\mathcal{L}) \to \mathrm{gr}\,H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G).$$

The first equality is a version of the Zassenhaus lemma. More generally, let F be a filtered quasicoherent sheaf on an algebraic variety S with a point $s \in S$. Then we have an isomorphism $\mathrm{gr}(F_s) \xrightarrow{\sim} (\mathrm{gr}\,F)_s$. In effect, the Rees construction associates to F a $\mathbb{G}_m$-equivariant quasicoherent sheaf RF on $S \times \mathbb{A}^1$, and $\mathrm{gr}\,F$ is the restriction of RF to $S \times \{0\}$. Hence both $\mathrm{gr}(F_s)$ and $(\mathrm{gr}\,F)_s$ are naturally isomorphic to the fiber of RF at $(s, 0) \in S \times \mathbb{A}^1$.

So it suffices to prove that $\mathrm{gr}\,\overline{\mathbf{z}}^*$ is surjective. But $\mathrm{gr}\,\overline{\mathbf{z}}^*[\mathcal{R}_\lambda] = (\mathbf{t}^{d_\lambda} + \text{lower}) \cap [\mathrm{Gr}_\lambda]$ where 'lower' stands for the terms in $\mathbb{C}[\mathbf{t}]^{W_\lambda}[\mathbf{t}]$ of degree lower

than d_λ in $\mathbf{t}$. Therefore $\operatorname{gr}\mathcal{L}$ is spanned by $\mathbf{t}^{-d_\lambda}[\mathcal{R}_\lambda]$ and $\mathbf{t}^{-d_\lambda}[\mathcal{R}_\lambda]$ is sent to $[\operatorname{Gr}_\lambda]$ at $\mathbf{t} = \infty$. $\qquad\square$

Remark 6.19. Thus we obtain a sheaf of algebras over $\mathbb{P}^1$ with coordinate $\mathbf{t}$. From the proof of Proposition 6.17, it is a sheaf of filtered algebras, and the associated graded sheaf is clearly flat, so the initial sheaf is flat as well. If we drop the loop rotation equivariance, we obtain a flat sheaf of commutative algebras; let $\mathcal{M}_C^{\mathbf{t}} \to \mathbb{P}^1$ denote its relative spectrum. The grading by degree of equivariant homology gives rise to an action of $\mathbb{G}_m$ on $\mathcal{M}_C^{\mathbf{t}}$ such that the projection $\mathcal{M}_C^{\mathbf{t}} \to \mathbb{P}^1$ is equivariant with respect to the natural action of $\mathbb{G}_m$ on $\mathbb{P}^1$. Hence all the fibers over $\mathbf{t} \neq 0, \infty$ are isomorphic.

Remark 6.20. It seems likely that when $\mathbf{N} = \mathfrak{g}$ is the adjoint representation, $H_*^{\mathbb{C}^\times \times G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\mathcal{R})$ is the spherical subalgebra in the graded Cherednik algebra (alias trigonometric DAHA) [OY16, 4.1], while $H_*^{G_{\mathcal{O}} \rtimes \mathbb{C}^\times}(\operatorname{Gr}_G)$ is the spherical subalgebra in the trigonometric Nil-DAHA (cf. [CF13, 1.2]). The above degeneration is related to the Inozemtsev degeneration [Ino89, Section 2] of the quantum Calogero-Moser system into the quantum Toda system.

Appendix A. A moduli space of solutions of Nahm's equations and homology of affine Grassmannian

We use mainly a compact Lie group G_c, rather than its complexification G in this section.

Let The Coulomb branch $\mathcal{M}_C$ of the gauge theory $G_c = \operatorname{SU}(k)$, $\mathbf{N} = 0$ is identified with M_k^0, the moduli space of centered monopoles by physical arguments ([SW97] for $k = 2$, [CH97] for arbitrary k).

Let us generalize this result to arbitrary compact Lie group G_c in our definition of $\mathcal{M}_C$. We shall use a moduli space of solutions of Nahm's equations, studied (in more general setting) by Bielawski [Bie97].

Let $G_c^\vee$ be the Langlands dual group of G_c. We consider Nahm's equations

$$\nabla_t T_\alpha + \frac{1}{2} \sum_{\beta,\gamma} \varepsilon_{\alpha\beta\gamma}[T_\beta, T_\gamma] = 0 \qquad (\alpha, \beta, \gamma = 1, 2, 3)$$

on the interval $(-1, 1)$, where T_α is a $\mathfrak{g}_c^\vee$-valued function. We require that T_α has at most simple poles at $t = \pm 1$. Then its residue $\operatorname{res}_{t=\pm 1} T_\alpha$ defines a Lie algebra homomorphism $\rho_c \colon \mathfrak{su}(2) \to \mathfrak{g}_c^\vee$. Then we further require that

ρ_c is the restriction of a homomorphism $\rho\colon \mathfrak{sl}(2) \to \mathfrak{g}^\vee$ such that $y = \rho(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$ is a principal nilpotent element in $\mathfrak{g}^\vee$.

We consider the moduli space of solutions, that is the quotient by the group $\mathcal{G}_{G_c^\vee}$ of gauge transformations $\gamma\colon [-1, 1] \to G_c^\vee$ such that $\gamma(\pm 1)$ is the identity element.

Theorem A.1. *The Coulomb branch $\mathcal{M}_C$ of the pure gauge theory for G_c (i.e., $\mathbf{N} = 0$) is the moduli space of solutions of Nahm's equations for the Langlands dual group $G_c^\vee$.*

This result immediately follows once we combine [BFM05, BF08] with [Bie97], as reviewed below.

Since the moduli space is a hyper-Kähler quotient, it is natural to expect that this remains true in physicists' (not yet mathematically precise) definition of $\mathcal{M}_C$.

While preparing the manuscript, we have noticed that this statement is mentioned in [Tel14, Remark 6.4].

Remark A.2. When $G_c = \mathrm{U}(k)$, the moduli space of solutions of Nahm's equations is isomorphic to the space of based maps from $\mathbb{P}^1$ to itself [Don84]. This description is the A_1 case of [Quiver, Theorem 3.1]: As for a quiver gauge theory of type ADE, the Coulomb branch $\mathcal{M}_C$ is the space of based maps from $\mathbb{P}^1$ to the flag variety of the corresponding type.

A.1. Homology of affine Grassmannian

When $\mathbf{N} = 0$, our proposal 3.13 states $\mathcal{M}_C$ is the spectrum of the equivariant Borel-Moore homology group $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ of the affine Grassmannian $\mathrm{Gr}_G = G_\mathcal{K}/G_\mathcal{O}$ of G, equipped with an algebra structure given by the convolution. Let us use a description which naturally arises from [BF08, Th. 3].

Let $G^\vee$ be the Langlands dual group and $\mathfrak{g}^\vee$ its Lie algebra. Let $\rho\colon \mathfrak{sl}(2) \to \mathfrak{g}^\vee$ be a Lie algebra homomorphism such that $y = \rho(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$ is a principal nilpotent element in $\mathfrak{g}^\vee$. Let $N^\vee$ be the unipotent subgroup of $G^\vee$ whose Lie algebra $\mathfrak{n}^\vee$ is the sum of negative eigenspaces of $h = \rho(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix})$. If we regard y as an element of $\mathfrak{n}^{\vee*}$ via an invariant pairing on $\mathfrak{g}^\vee$, it is stabilized by $N^\vee$.

The cotangent bundle $T^*G^\vee = G^\vee \times \mathfrak{g}^\vee$ is a holomorphic symplectic manifold with a $G^\vee \times G^\vee$-action by left and right multiplication. Here we identify $\mathfrak{g}^{\vee*}$ with $\mathfrak{g}^\vee$ by the invariant inner product. We have a (complex) moment map $\mu_\mathbb{C}\colon G^\vee \times \mathfrak{g}^\vee \to \mathfrak{g}^{\vee*} \oplus \mathfrak{g}^{\vee*}$ given by $\mu_\mathbb{C}(g, \xi) = (\xi, -\mathrm{Ad}(g^{-1})\xi)$

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for $g \in G^\vee$, $\xi \in \mathfrak{g}^\vee$. Let $\overline{\mu}_\mathbb{C} \colon G^\vee \times \mathfrak{g}^\vee \to \mathfrak{n}^{\vee *} \oplus \mathfrak{n}^{\vee *}$ be the moment map for the $N^\vee \times N^\vee$-action, that is the composite of $\mu_\mathbb{C}$ and the natural projection $\mathfrak{g}^{\vee *} \oplus \mathfrak{g}^{\vee *} \to \mathfrak{n}^{\vee *} \oplus \mathfrak{n}^{\vee *}$. Now [BF08, Th. 3] implies

$$(A.3) \qquad \operatorname{Spec}(H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)) \cong \overline{\mu}_\mathbb{C}^{-1}(y, y)/N^\vee \times N^\vee.$$

Remark A.4. In [BF08, Th. 3], $H_*^{G_\mathcal{O} \rtimes \mathbb{C}^\times}(\mathrm{Gr}_G)$ is described as a quantum hamiltonian reduction of the ring of differential operators on $G^\vee$. The above description is its classical limit. On the other hand, $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$ is described as $3_{\mathfrak{g}^\vee}^{G^\vee}$ in the earlier paper [BFM05, Th. 2.12] as we mentioned in §3(x)(a). Since both are $H_*^{G_\mathcal{O}}(\mathrm{Gr}_G)$, we have an isomorphism $\overline{\mu}_\mathbb{C}^{-1}(y, y)/N^\vee \times N^\vee \cong 3_{\mathfrak{g}^\vee}^{G^\vee}$. While preparing the manuscript, we have learned that an explicit construction of the isomorphism is given in [Tel14, Th. 6.3].

A.2. Moduli space of solutions of Nahm's equations

Let $M_{G_c^\vee}$ be the moduli space in Theorem A.1. By [Bie97] it is a submanifold of $T^*G^\vee$ defined as follows. Let $\rho \colon \mathfrak{sl}(2) \to \mathfrak{g}^\vee$ as above. We define Kostant-Slodowy slice $S(\rho) = y + Z(x)$, where $x = \rho(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix})$, $y = \rho(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix})$ and $Z(x)$ is the centralizer of x in $\mathfrak{g}^\vee$. Now [Bie97, Cor. 4.1] states $M_{G_c^\vee} = \mu_\mathbb{C}^{-1}(S(\rho) \times S(\rho))$.

Let us further rewrite $M_{G_c^\vee}$ as a holomorphic symplectic quotient of $T^*G^\vee$. The result is implicit in [Bie97, §3]. The point is that $S(\rho)$ equals to $\nu^{-1}(y)/N^\vee$, where $N^\vee$ is the unipotent subgroup of $G^\vee$ as above, and $\nu \colon \mathfrak{g}^{\vee *} \to \mathfrak{n}^{\vee *}$ is the natural projection. Since $\overline{\mu}_\mathbb{C} = (\nu \times \nu) \circ \mu_\mathbb{C}$, we have $M_{G_c^\vee} \cong \overline{\mu}_\mathbb{C}^{-1}(y, y)/N^\vee \times N^\vee$.

Combining with (A.3), we finish the proof of Theorem A.1.

A.3. Centered SU(2)-monopoles

In the remainder of this section, we explain that Theorem A.1 reproduces [SW97, CH97] for $G_c = \mathrm{SU}(k)$.

Let M_k be the framed moduli space of charge k, $\mathrm{SU}(2)$-monopoles on $\mathbb{R}^3$ (see [AH88]). By [Hit83], it is naturally bijective to the moduli space of solutions of Nahm's equations with values in $\mathfrak{u}(k)$ on the interval $(-1, 1)$ such that T_α has at most simple poles at $t = \pm 1$ and its residue $\mathrm{res}_{t=\pm 1} T_\alpha$ gives an irreducible k-dimensional representation ρ of $\mathrm{SU}(2)$. They are even isomorphic as hyper-Kähler manifolds [Nak93]. Since the irreducible representation corresponds to the principal nilpotent for $\mathfrak{gl}(k)$, we conclude $M_k = M_{\mathrm{U}(k)}$. We identify M_k with the moduli space of solutions of Nahm's equations for $G_c = \mathrm{U}(k)$ hereafter.

We need to modify this description in the case of centered monopoles. Let $\tilde{M}_k$ be the space of solutions, and $\mathcal{G}_k = \mathcal{G}_{\mathrm{U}(k)}$ be the group of gauge transformations, i.e., the space of maps $\gamma\colon [-1,1] \to \mathrm{U}(k)$ such that $\gamma(\pm 1) = 1$. We have $M_k = \tilde{M}_k/\mathcal{G}_k$. We introduce a larger group $\tilde{\mathcal{G}}_k$ consisting of maps γ, for which we require $\gamma(-1) = 1$ and $\gamma(1)$ commutes with the pole α of T_α for $\alpha = 1, 2, 3$. Since $\mathrm{res}\, T_\alpha$ gives an irreducible representation, $\gamma(1)$ must be scalar. Moreover, the action of $\tilde{\mathcal{G}}_k$ is still free by the irreducibility of (∇_t, T_α). We have a free action of $\tilde{\mathcal{G}}_k/\mathcal{G}_k = \mathrm{U}(1)$ on M_k.

We also have an $\mathbb{R}^3$-action on M_k given by $T_\alpha \mapsto T_\alpha + ix_\alpha$ ($x_\alpha \in \mathbb{R}$, $\alpha = 1, 2, 3$). The quotient

$$M_k^0 = (M_k/\mathrm{U}(1))/\mathbb{R}^3$$

is called the moduli space of centered monopoles. In [AH88], this space was introduced in terms of monopoles, one can check that it is equivalent to the above definition. The detail is left as an exercise for the reader.

Choose a trivialization of the vector bundle, and write $\nabla = d + T_0$. By the gauge transformation

$$\gamma(t) \stackrel{\mathrm{def.}}{=} \exp\left(\frac{\mathrm{id}}{k}\int_{-1}^{t}\mathrm{tr}\, T_0(s)ds\right) \in \tilde{\mathcal{G}}_k,$$

we can make $\mathrm{tr}\, T_0 = 0$. The Nahm's equation implies that $\frac{d}{dt}\mathrm{tr}\, T_\alpha = 0$, hence we can make $\mathrm{tr}\, T_\alpha = 0$ by the $\mathbb{R}^3$-action. Therefore

$$M_k^0 = \{(T_0, T_\alpha)\colon (-1,1) \to \mathfrak{su}(k)\}/\tilde{\mathcal{G}}_k',$$

where $(d + T_0, T_\alpha)$ satisfies Nahm's equation and the condition of the pole. The group $\tilde{\mathcal{G}}_k'$ is the subgroup of $\tilde{\mathcal{G}}_k$ preserving the condition $\mathrm{tr}\, T_0 = 0$. Therefore it consists of maps $\gamma\colon [-1,1] \to \mathrm{SU}(k)$ with $\gamma(-1) = 1$, $\gamma(1) \in \mathbb{Z}_k = \mathrm{U}(1) \cap \mathrm{SU}(k)$.

Since T_0, T_α are $\mathfrak{su}(k)$-valued, we can replace $\tilde{\mathcal{G}}_k'$ by the space of maps $\gamma\colon [-1,1] \to \mathrm{SU}(k)/\mathbb{Z}_k$. Moreover as $[-1,1]$ is contractible, such γ lifts uniquely to $\mathrm{SU}(k)$ when we set $\gamma(-1) = 1$. Therefore the group is unchanged under this replacement. Thus $\tilde{\mathcal{G}}_k' = \mathcal{G}_{\mathrm{SU}(k)/\mathbb{Z}_k}$, where $\mathcal{G}_{\mathrm{SU}(k)/\mathbb{Z}_k}$ is the space of maps $\gamma\colon [-1,1] \to \mathrm{SU}(k)/\mathbb{Z}_k$ such that $\gamma(\pm 1) = 1$. Thus we see that M_k^0 is nothing but $M_{\mathrm{SU}(k)/\mathbb{Z}_k}$.

The Langlands dual group of $\mathrm{SU}(k)$ is $\mathrm{SU}(k)/\mathbb{Z}_k$, hence $\mathcal{M}_C$ in Theorem A.1 is the moduli space M_k^0 of centered $\mathrm{SU}(2)$-monopoles with charge k, as given in [SW97, CH97].

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